

The geometric structure of class two nilpotent groups and subgroup growth

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Abstract

In this paper we derive an explicit expression for the normal zeta function of class two nilpotent groups whose associated Pfaffian hypersurface is smooth. In particular, we show how the local zeta function depends on counting \mathbb{F}_p -rational points on related varieties, and we describe the varieties that can appear in such a decomposition. As a corollary, we also establish explicit results on the degree of polynomial subgroup growth in these groups, and we study the behaviour of poles of this zeta function. Under certain geometric conditions, we also confirm that these functions satisfy a functional equation.

1 Introduction

Zeta functions of a finitely generated group G were introduced in [12] as a non-commutative analogue to the Dedekind zeta function of a number field. They are used to study the arithmetic and asymptotic properties of the sequence of numbers

$$a_n(G) = |\{H \leq_f G : |G : H| = n\}|.$$

The fact that G is finitely generated ensures that $a_n(G)$ is finite for all n . The term *subgroup growth* is used to describe the study of the sequences $a_n(G)$ and $s_n(G) = \sum_{i=1}^n a_i(G)$.

We can also consider different variants of subgroup growth, for example, we may only count the normal subgroups of finite index. To do this we define the sequence

$$a_n^\triangleleft(G) = |\{H \triangleleft_f G : |G : H| = n\}|.$$

Or we can define a sequence

$$a_n^\wedge(G) = |\{H \leq_f G : |G : H| = n, \hat{H} \cong \hat{G}\}|,$$

to count those subgroups whose profinite completion is isomorphic to the profinite completion of the group itself. We define the *zeta function* of G to be the formal Dirichlet series

$$\zeta_G^*(s) = \sum_{n=1}^{\infty} a_n^*(G) n^{-s},$$

where $*$ $\in \{\leq, \triangleleft, \wedge\}$, and s is a complex variable. The abscissa of convergence of $\zeta_G^*(s)$, which we denote by α_G^* , determines *the degree of polynomial subgroup growth* since

$$\alpha_G^* := \inf\{\alpha \geq 0 : \text{there exists } c > 0 \text{ with } s_n^*(G) < cn^\alpha \text{ for all } n\}.$$

When G is a finitely generated torsion-free nilpotent group, called henceforth a \mathfrak{T} -group, the function $a_n^*(G)$ grows polynomially and is multiplicative, in the sense that $a_n^*(G) = \prod_i a_{p_i^{k_i}}^*(G)$ where $n = \prod_i p_i^{k_i}$. Therefore, like the zeta functions in number theory, we can write the Dirichlet series as an Euler product decomposition of *the local zeta functions*, i.e.,

$$\zeta_G^*(s) = \prod_{p \text{ prime}} \zeta_{G,p}^*(s),$$

where

$$\zeta_{G,p}^*(s) = \sum_{k=0}^{\infty} a_{p^k}^*(G) p^{-ks}$$

counts only the subgroups with p -power index.

A fundamental theorem of Grunewald, Segal and Smith [12] states that these local zeta functions of \mathfrak{T} -groups are rational functions in p^{-s} . This rationality implies that the coefficients $a_n^*(G)$ behave smoothly; in particular, they satisfy a linear recurrence relation. This is proved by expressing the zeta function as a p -adic integral and then using a model theoretic black box to deduce the rationality. Results of Denef on p -adic integrals arising from a local Igusa zeta function have been generalised by Grunewald and du Sautoy in [9] and by Voll [19]. This has lead to some new interesting problems. For example it transpires that the Hasse-Weil zeta function of smooth projective varieties is possibly a better analogue for the zeta functions in the group setting than the Dedekind zeta function of number fields, as previously mentioned. This initiated an arithmetic-geometric approach to zeta functions of groups. Theorem 1.6 in [9] gives an explicit decomposition for the zeta function as a sum of rational functions $P_i(p, p^{-s})$, with coefficients coming from counting points mod p on various varieties. In a subsequent paper [8]

du Sautoy presents a group whose zeta function depends on the \mathbb{F}_p -rational points on an elliptic curve. The explicit zeta function $\zeta_{G,p}^{\mathfrak{A}}(s)$ of this elliptic curve example is calculated by Voll in [21], and he also proves that it satisfies a functional equation. Since the zeta function depends on counting the number of \mathbb{F}_p -points on the elliptic curve, an expression that is more complicated than just a polynomial in p , the proof that it satisfies a functional equation uses the fact that the Hasse-Weil zeta function of smooth projective varieties also satisfies a functional equation. In [22], he extends this to all smooth hypersurfaces, with the restriction that these hypersurfaces should not contain any lines.

In the current paper, we generalise Voll's work by removing the condition on linear subspaces. We still require that the hypersurface associated with the groups is smooth. The main contribution of this generalisation is that it sheds some light on which varieties can arise in the context of zeta functions of groups. In particular, we can describe the varieties and the numerical data associated to the poles of the zeta function quite explicitly.

Let us now define the particular varieties that will be of interest to us.

Definition 1.1. Let Γ be a class 2 \mathfrak{T} -group with $Z(\Gamma) = [\Gamma, \Gamma]$, and a presentation of the form

$$\Gamma := \langle x_1, \dots, x_d, y_1, \dots, y_{d'} : [x_i, x_j] = M(\mathbf{y})_{ij} \rangle,$$

where d is even and $M(\mathbf{y})$ is the matrix of relations; an antisymmetric $d \times d$ matrix with entries that are linear forms in the y_i . We call the variety defined by the equation

$$\mathfrak{P}_{\Gamma} : \det^{\frac{1}{2}} M(\mathbf{y}) = 0$$

the *Pfaffian hypersurface associated to Γ* . We exclude from this definition the case when the determinant is identically zero.

We shall extend this definition for a general class 2 \mathfrak{T} -group G by writing $G = \Gamma \times \mathbb{Z}^m$, where Γ is as above and $m \geq 0$. Then $\mathfrak{P}_G := \mathfrak{P}_{\Gamma}$.

Note that the condition that d is an even number is necessary to ensure that the corresponding Pfaffian hypersurface is non-trivial. Similarly the extended definition, we want to exclude the \mathbb{Z}^m , because otherwise the Pfaffian would be identically zero.

Definition 1.2. The set of $(k-1)$ -planes in $\mathbb{P}^{d'-1}$ forms a variety called the Grassmannian, denoted by $\mathbb{G}(k-1, d'-1)$. The affine analogue is the set of k -planes in affine d' -dimensional space, which we denote by $G(k, d')$.

It is well-known that $\mathbb{G}(k-1, d'-1)$ can be embedded in \mathbb{P}^N via the Plücker embedding, where $N = \binom{n}{k}$.

Definition 1.3. The Fano variety associated to a hypersurface $X \subset \mathbb{P}^{d'-1}$ is the variety

$$F_{k-1}(X) = \{\Pi \in \mathbb{G}(k-1, d'-1) : \Pi \subset X\}$$

of $(k-1)$ -planes contained in X .

By abuse of notation we will identify $(k-1)$ -dimensional linear spaces $\Pi \in \mathbb{P}^{d'-1}$ with the corresponding point $\Pi \in \mathbb{G}(k-1, d'-1)$.

We shall show in this paper that we can derive a decomposition of the normal local zeta function of G , which is similar to that of Theorem 1.6 in [9], and also describe the varieties that appear in our decomposition exactly as the Pfaffian hypersurface, and components of the Fano varieties of the linear subspaces contained on the Pfaffian. Here components may arise because the Fano variety may be reducible and, moreover, since we consider determinantal varieties, the rank of the defining matrix needs to be taken into account, and different components may give different ranks.

The local zeta functions of finitely generated torsion-free nilpotent groups calculated over the years, see [7] for a collection of these, exhibit curious symmetries which suggest the possibility all zeta functions of groups may have a similar local functional equation. In a recent paper [19], Voll has shown that a functional equation is satisfied for local zeta functions of \mathfrak{T} -groups counting all subgroups, conjugacy classes of subgroups and representations. However, for the local normal zeta function Voll's proof works only in nilpotency class two. Explicitly, let $G = \Gamma \times \mathbb{Z}^m$ be a class two nilpotent group, where $Z(\Gamma) = [\Gamma, \Gamma]$ with $h(\Gamma^{ab}) = d$ and $h(Z(\Gamma)) = d'$, which yields $h(G) = d + d' + m$, $h(Z(G)) = m + d'$, $h([G, G]) = d'$, $h(G/[G, G]) = m + d$ and $h(G/Z(G)) = d$. Here h is the Hirsch length, which for a finitely generated group G is the number of infinite cyclic factors in a subnormal series of G . Then, as Voll shows, the functional equation in the class two is of the form

$$\zeta_{G,p}^{\triangleleft}(s)|_{p \mapsto p^{-1}} = (-1)^{d+d'+m} p^{\binom{d+d'+m}{2} - (2d+d'+m)s} \zeta_{G,p}^{\triangleleft}(s).$$

Examples of local normal zeta functions of groups of nilpotency class three calculated by Woodward [7] show that a functional equation is not always satisfied. However, [7] contains also a number of examples of local normal zeta functions in higher classes where the functional equation holds. It remains mysterious when there is a functional equation for the normal zeta function and when not.

Other type of results on functional equations connected to zeta functions of groups include work by Lubotzky and du Sautoy [11] on zeta functions counting those subgroups whose profinite completion is isomorphic to the profinite completion. These results have been partially generalised by Berman [4].

In the current paper we confirm with a different method than Voll's that the functional equation is satisfied in the case where the Pfaffian hypersurface

associated to the group, and the components of its Fano varieties, are smooth and absolutely irreducible. This is the best possible result using our methods, since the final step uses the fact that the Hasse-Weil zeta function of points on a smooth and absolutely irreducible algebraic variety satisfies a functional equation.

Our main theorem also has applications to subgroup growth. In particular, we provide evidence that some of the best bounds known so far for subgroup growth may be exact. We also deduce results on the number and nature of poles of the normal zeta function for class two nilpotent groups.

We refer the reader to [10] for the most recent survey of the general theory of these zeta functions.

1.1 Results

Before stating our main results, we begin with a few preliminary definitions.

A classical result, e.g. [12] states that the zeta function of the free abelian group of rank d can be expressed in terms of $\zeta(s)$, the Riemann zeta function, as follows:

$$\zeta_{\mathbb{Z}^d}(s) = \zeta(s)\zeta(s-1)\dots\zeta(s-(d-1)).$$

We write $\zeta_p(s) = \frac{1}{1-p^{-s}}$ for its local factors, where p is a prime.

Definition 1.4. A *flag* of type I in $V = \mathbb{F}_p^{d'}$, where $I = \{i_1, \dots, i_l\}_< \subseteq \{1, \dots, d'\}$, (where $\{i_1, \dots, i_l\}_<$ is an indexing set ordered such that $i_1 < i_2 < \dots < i_l$) is a sequence $(V_{i_j})_{i_j \in I}$ of incident vector spaces

$$\{0\} < V_{i_1} < \dots < V_{i_l} \leq V$$

with $\dim_{\mathbb{F}_p}(V_{i_j}) = i_j$. The flags of type I form a projective variety \mathcal{F}_I , whose number of \mathbb{F}_p -points is denoted by $b_I(p) \in \mathbb{Z}[p]$, where

$$b_I(p) = \binom{d'}{i_l} \binom{i_l}{i_{l-1}} \dots \binom{i_3}{i_2} \binom{i_2}{i_1}.$$

Definition 1.5. The *Igusa factor* is a rational function in the set of variables $\mathbf{U} = \{U_1, \dots, U_n\}$ defined by

$$I_n(\mathbf{U}) = \sum_{I \subseteq \{1, \dots, n\}} b_I(p^{-1}) \prod_{i \in I} \frac{U_i}{1 - U_i},$$

where $b_I(p)$ is the number of flags of type I in \mathbb{F}_p^{n+1} .

Definition 1.6. We call a rational function of the form

$$E_{\iota_i}(X_i, Y_{\iota_i}) = \frac{p^{-d_{\iota_i}} Y_{\mathbb{B}_i} - p^{-n_i} X_i}{(1 - X_i)(1 - Y_{\mathbb{B}_i})}$$

the \mathbb{B}_i^{th} *exceptional factor* with numerical data n_i and $d_{\mathbb{B}_i}$.

Our main theorem is the following

Theorem 1.7. *Let $G = \Gamma \times \mathbb{Z}^m$ be a class two nilpotent group such that $Z(\Gamma) = [\Gamma, \Gamma]$ and $d = h(G/Z(G))$ is an even number. Set $d' = h([G, G])$ and let \mathfrak{P}_G be the associated Pfaffian hypersurface, which we assume to be reduced, smooth and non-zero. Then for almost all primes p*

$$\zeta_{G,p}^{\triangleleft}(s) = \zeta_{\mathbb{Z}^{d+m},p}(s) \zeta_p((d+d')s - d(d+m))(W_0(\mathbf{X}, \mathbf{Y}) + \sum_{i=1}^k \sum_{\mathbb{B}_i} \mathbf{n}_{\mathbb{B}_i}(p) \delta_{\mathbb{B}_i} W_{\mathbb{B}_i}(\mathbf{X}, \mathbf{Y})),$$

where the inner sum is over components $F_{\mathbb{B}_i}$ of $F_{i-1}(\mathfrak{P}_G)$, and $\delta_{\mathbb{B}_i}$ is the Kronecker delta function. The $\mathbf{n}_{\mathbb{B}_i}(p)$ denotes the number of \mathbb{F}_p -rational points on these components, and

$$W_{\mathbb{B}_i}(\mathbf{X}, \mathbf{Y}) = I_{d'-i-1}(X_{d'-1}, \dots, X_{i+1}) E_{\mathbb{B}_i}(X_i, Y_{\mathbb{B}_i}) I_{i-1}(Y_{\mathbb{B}_i-1}, \dots, Y_{\mathbb{B}_1}).$$

Here I_n is an Igusa factor with n variables and we set $E_0 = I_0 = I_{-1} = 0$. The numerical data is

$$\begin{aligned} X_i &= p^{i(d+d'+m-i)-(d+i)s}, i \geq 1 \\ Y_1 &= p^{(d+m+d_1)-(d-1)s}, \\ Y_{\mathbb{B}_i} &= p^{(i(d+m)+d_{\mathbb{B}_i})-(d+i-1)s}, i \geq 2. \end{aligned}$$

where $d_{\mathbb{B}_i}$ is the dimension of $F_{\mathbb{B}_i}$.

Remark 1.8. When $\delta_{\mathbb{B}_i}$ is 1 or 0 is a technical condition, that depends on the possible reducibility of the Fano variety and the rank of the matrix defining the Pfaffian hypersurface. We will delay the exact definition to Remark 6.7.

Remark 1.9. Lemma 2.1 below gives a generic bound $k \leq 5$. This gives us also a good bound for the number of poles as in Corollary 1.14.

In all the following Corollaries, we will assume G satisfies the conditions of the main theorem. Moreover, in the results about the degree of polynomial subgroup growth, we have to assume that the Pfaffian hypersurface associated to G behaves as a generic hypersurface of degree $\frac{d}{2}$. This will be explained in Section 2.

Corollary 1.10. *The abscissa of convergence is*

$$\alpha_G^{\triangleleft} = \max_{1 \leq i \leq d'-1} \left\{ d + m, \frac{i(d + d' + m - i) + 1}{(d + i)} \right\}. \quad (1)$$

Note that this is the lower limit given in [18, Theorem 1.3], except that we have removed the restriction on $Z(G) = [G, G]$.

The next corollary follows immediately from Corollary 1.10.

Corollary 1.11. *Let G be a class two nilpotent group with $d' = h([G, G]) \leq 6$, such that the Pfaffian hypersurface associated to G is a smooth generic hypersurface. Then $\alpha_G^\Delta = d + m$.*

The proof of Corollary 1.11 uses the fact that a generic hypersurface is smooth in $\mathbb{P}^{d'-1}$, if $d' \leq 6$ (see Remark 8.3 in [2]), and that the maximum in (1) is attained at $d + m$ if $4d(d + m) > d'^2$.

In number theory there are much finer asymptotic estimates concerning the asymptotic behaviour of the coefficients of a Dirichlet series; these are the so-called Tauberian theorems. To apply these results we need to be able to analytically continue the zeta function to the left of its abscissa of convergence. The analytic continuation by some $\varepsilon > 0$ for zeta functions of \mathfrak{T} -groups is proved in [9] using the analytic continuation of the Artin L -functions.

Using the combination of Theorem 1.7 and the appropriate Tauberian theorems it is possible to obtain more detailed results on subgroup growth.

Corollary 1.12. *Let $\beta_G^\Delta \in \mathbb{N}$ be the multiplicity of the pole of the zeta function located at the abscissa of convergence. Then exists $c_G \in \mathbb{R}$ such that*

$$s_n^\Delta(G) \sim c_G \cdot n^{\alpha_G^\Delta} (\log n)^{\beta_G^\Delta - 1}.$$

as $n \rightarrow \infty$.

Remark 1.13. It is clear from the statement of Theorem 1.7 that all the poles are simple, except when the numerical data is such that it produces a multiple pole. In particular, we have $\beta_G^\Delta = 1$.

Theorem 1.7 indicates that the existence of a multiple pole is just a coincidence of the numerical data, rather than any structural properties of the function or the group. It should be noted that these coincidences exist for smooth Pfaffians. Moreover, it is not difficult to prove that certain types of singularities on the Pfaffian always produce genuine multiple poles, for instance the ordinary double point at the origin, as is the case in the Pfaffian hypersurface for the group

$$U_3 := \langle x_1, \dots, x_4, y_1, y_2, y_3 : [x_1, x_2] = y_1, [x_2, x_3] = y_2, [x_3, x_4] = y_3 \rangle,$$

produces a double pole in the zeta function.

Our final corollary bounds the number of poles for all these zeta functions. This follows immediately from Theorem 1.7.

Corollary 1.14. *Let G be a group as in Theorem 1.7, and assume that the Pfaffian is a generic hypersurface of degree $\frac{d}{2}$. Then the number of poles of the local zeta function $\zeta_{G,p}^\Delta(s)$ is at most $d + d' + m + r$ where r is the number of irreducible components of the Fano varieties on the Pfaffian hypersurface.*

Finally, we observe the functional equations, proved by Voll [19].

Corollary 1.15. *Assume the Pfaffian hypersurface of G is absolutely irreducible, and that the components of the Fano varieties appearing in the decomposition of the zeta function of G are smooth and absolutely irreducible. Then the local normal zeta function satisfies a functional equation of the form*

$$\zeta_{G,p}^{\triangleleft}(s)|_{p \mapsto p^{-1}} = (-1)^{d+d'+m} p^{\binom{d+d'+m}{2} - (2d+d'+m)s} \cdot \zeta_{G,p}^{\triangleleft}(s).$$

It would be interesting to know if all the components of the Fano varieties corresponding to a smooth irreducible Pfaffian have the same dimension. For quadrics this is known, also for the Fano variety of lines on a cubic hypersurface (see [1]). However, for singular Pfaffians this is not true. In [5] Browning and Heath-Brown give an example of a cubic hypersurface defined by the equation $y_1 y_2 y_3 + y_1^2 y_4 + y_2^2 y_5 = 0$, whose Fano variety of planes consists of \mathbb{P}^1 and a point. This hypersurface can be encoded as the Pfaffian of the group

$$G = \begin{matrix} < x_1, \dots, x_6, y_1, \dots, y_5 : [x_1, x_4] = y_1, [x_1, x_5] = y_2, [x_2, x_4] = y_5, \\ [x_2, x_5] = y_3, [x_2, x_6] = y_4, [x_3, x_5] = y_1, [x_3, x_6] = y_2 > . \end{matrix}$$

However, this Pfaffian hypersurface is not smooth and hence our theorem does not apply to this group. The splitting of the Fano variety of planes in this case is curious and deserves some further investigation.

1.2 Layout of the paper

Our results on subgroup growth and abscissae of convergence, namely Corollaries 1.10-1.14, are proved in Section 2. In Section 3 we briefly comment on how the functional equation described in Corollary 1.15 can be deduced from the main theorem.

Next in Section 4 we explain how to enumerate lattices in \mathbb{Z}_p^m , and calculate the number of lattices of a fixed elementary divisor type. In Section 5 we define Grassmannian and flag varieties, give coordinates for the Plücker embedding of the Grassmannian and define flags in terms of these coordinates. We also determine when two different lattices lift the same flag. In Section 6 we shall discuss solution sets of systems of linear congruences, define Pfaffian hypersurfaces and give a careful description of the geometry needed to find a solution set of general congruences. In Section 7 we compute the p -adic valuations which arise in the solution sets.

In the Sections 8 to 10 we decompose the zeta function and apply our earlier work to complete the proof of Theorem 1.7. Section 11 is an explicit example and illustration of the general theory.

2 Abscissae of convergence

In this section we prove the Corollary 1.10 on the rate of subgroup growth and prove a result about the number of poles of the zeta function in the case that the Pfaffian hypersurface is generic; generic meaning that, there is a non-empty open subset U of the parameter space of hypersurfaces such that for every point in U the answer to the property we are asking, is uniform.

Our main theorem, Theorem 1.7, is stated in terms of local zeta functions. However, the degree of polynomial subgroup growth is equal to the abscissa of convergence of the global zeta function. Thus we need to analyse the convergence of an infinite product. This can be done using the following well-known results.

(A) An infinite product $\prod_{n \in J} (1 + a_n)$ converges absolutely if and only if the corresponding sum $\sum_{n \in J} |a_n|$ converges.

(B) $\sum_p |p^{-s}|$ converges at $s \in \mathbb{C}$ if and only if $\Re(s) > 1$.

In view of (B), $\frac{1}{1-p^{-a_i s + b_i}}$ has abscissa of converge $\frac{b_i}{a_i}$, so (A) implies that $\prod_p \text{prime} \frac{1}{1-p^{-a_i s + b_i}}$ has abscissa of convergence $\frac{b_i + 1}{a_i}$.

If we can show that the abscissa of convergence is determined by the denominator of the zeta function, then we can use the above observation to deduce the desired result, namely Corollary 1.10.

First, we note that the Igusa-type factors in the numerator do not affect the abscissa of convergence. Indeed, they are a combination of the numerical data $p^{-a_i s + b_i}$ from the denominator, multiplied with sums of powers of p^{-1} which come from the factors $b_I(p^{-1})$. In particular, all the Igusa type of factors in the numerator will converge no worse than the abscissa of convergence coming from the denominator.

The second main ingredient in the explicit formula in the statement of the Theorem 1.7 are the β_i^{th} *exceptional factors* in variables X_i and Y_{β_i} , which are of the form

$$E_{\beta_i}(X_i, Y_{\beta_i}) = \frac{p^{-d_{\beta_i}} Y_{\beta_i} - p^{-n_i} X_i}{(1 - X_i)(1 - Y_{\beta_i})}.$$

To analyse this factor we need to take into account the numbers $\mathbf{n}_{\beta_i}(p)$. By the Lang-Weil estimates [16], we have $\mathbf{n}_{\beta_i}(p) = \delta p^{d_{\beta_i}} + O(p^{d_{\beta_i} - \frac{1}{2}})$, where $\delta = (\frac{d}{2} - 1)(\frac{d}{2} - 2)$. We can rewrite the numerator of the exceptional factor as $p^{-d_{\beta_i}} Y_{\beta_i} - p^{-n_i} X_i = p^{-d_{\beta_i}} (Y_{\beta_i} - p^{-c_{\beta_i}} X_i)$, so the numerator of $\mathbf{n}_{\beta_i}(p) E_{\beta_i}(X_i, Y_{\beta_i})$ is equal to $\delta Y_{\beta_i} - \delta p^{-c_{\beta_i}} X_i + O(p^{-\frac{1}{2}} Y_{\beta_i}) + O(p^{-c_{\beta_i} - \frac{1}{2}} X_i)$ and this will converge no worse than the denominator $(1 - X_i)(1 - Y_{\beta_i})$ for large enough p .

This gives us a first estimate for the abscissa of convergence just by recording the numerical data from the poles, namely

$$\alpha_G^{\triangleleft} = \max_{2 \leq i \leq d' - 1} \left\{ d + m, \frac{i(d + d' + m - 1) + 1}{d + i}, \frac{i(d + m) + d_{\beta_i} + 1}{d + i - 1}, \frac{d + m + d_1 + 1}{d - 1}, \frac{d + d' + m}{d + 1} \right\}. \quad (2)$$

In order to finish the proof of the Corollary 1.10 we need to estimate the $d_{\mathcal{B}_i}$ appearing in the numerical data.

We can make some preliminary observations and reductions. In particular, it suffices to consider only the cases where $d \geq 6$. Indeed, if $d = 2$ then we may assume G is the Heisenberg group and this is known to have a normal zeta function with abscissa of convergence 2, see [12]. In the case $d = 4$, the Pfaffian hypersurface is a quadric, and the smooth projective quadrics over \mathbb{F}_p have been classified, see e.g. [14]. By inspecting an explicit list we can use Theorem 1.7 to deduce that the abscissa of convergence is always 4. We also note that $d' \leq \frac{d(d-1)}{2}$, with equality for the free class two nilpotent groups. Trivially we have $1 \leq i \leq d'$.

If we assume that the Pfaffian hypersurface of degree $\frac{d}{2}$ behaves like a generic hypersurface in $\mathbb{P}^{d'-1}$, then the dimension of the Fano variety of $(i-1)$ -planes is

$$d_{\mathcal{B}_i} = i(d' - i) - \binom{\frac{d}{2} + (i-1)}{i-1}, \quad (3)$$

for $\frac{d}{2} \geq 3$ (see e.g. [13, Theorem 12.8]). Note that if $d_{\mathcal{B}_i} < 0$, then the Fano variety of $(i-1)$ -planes on the Pfaffian is empty.

Lemma 2.1. *Let \mathfrak{P}_G be a smooth and generic Pfaffian hypersurface associated to a group G as in Theorem 1.7. Then $F_4(\mathfrak{P}_G)$ is the highest non-empty Fano variety.*

Proof. We need to determine for which d, d', i we have

$$d_{\mathcal{B}_i} = i(d' - i) - \binom{\frac{d}{2} + (i-1)}{i-1} \geq 0,$$

under the additional hypothesis $d \geq 6$, $d' \leq \frac{d(d-1)}{2}$ and $1 \leq i \leq d'$. These additional conditions are all trivial bounds coming from the group structure that governs the Pfaffian hypersurface. In general the binomial term grows in d^{i-1} , while d' grows at most in d^2 , so asymptotically the binomial term surpasses the quadratic term quickly. In order to make this precise for small d and d' . We first observe that

$$i(d' - i) - \binom{\frac{d}{2} + (i-1)}{i-1} \geq 0$$

if and only if

$$d' \geq \frac{1}{i} \binom{\frac{d}{2} + (i-1)}{i-1} + i.$$

Trivially, we have $d' \leq \frac{d(d-1)}{2}$ so it is enough to show that

$$\frac{d(d-1)}{2} \geq \frac{1}{i} \binom{\frac{d}{2} + (i-1)}{i-1} + i$$

fails for all large enough d and i . It is easy to calculate that if $i = 6$ then this inequality holds only if $d \leq 5$, which means that $d \leq 4$ and we have already dealt with this case. Thus it is enough to consider $i \leq 5$, as claimed \square

For hypersurfaces that are assumed to be smooth, but not necessarily generic a proposition of Starr from the Appendix of [5] states that $F_{i-1}(\mathfrak{P}_G) = \emptyset$ for $i > \frac{d'}{2}$. Beheshti [3] has given some bounds for these dimensions in characteristic zero. However, we do not know enough about these dimensions in characteristic p and so we only consider generic hypersurfaces.

We have from (2)

$$\alpha_G^\triangleleft = \max_{2 \leq i \leq d'-1} \left\{ d + m, \frac{i(d + d' + m - i) + 1}{(d + i)}, \frac{i(d + m) + d_{\mathfrak{B}_i} + 1}{d + i - 1}, \frac{d + m + d_1 + 1}{d - 1}, \frac{d + d' + m}{d + 1} \right\}.$$

We shall prove the corollary in two stages, first when $4d(d + m) > d'^2$ and then $4d(d + m) \leq d'^2$. In the first case $4d(d + m) > d'^2$ and generalising the results from [18] that

$$\max_{1 \leq i \leq d'-1} \left\{ d + m, \frac{i(d + d' + m - i) + 1}{d + i} \right\} = d + m.$$

It is an easy case analysis to show that

$$\alpha_G^\triangleleft = \max_{2 \leq i \leq 5} \left\{ d + m, \frac{i(d + m) + d_{\mathfrak{B}_i} + 1}{d + i - 1}, \frac{d + d' + m}{d + 1} \right\} = d + m,$$

using the estimate for the dimension and the inequality $4d(d + m) > d'^2$, and we leave this to the reader.

In the second case, $4d(d + m) \leq d'^2$. We again leave $i = 1$ to the reader, and we have from (2)

$$\alpha_G = \max_{2 \leq i \leq d'-1} \left\{ \frac{i(d + d' + m - i) + 1}{(d + i)}, \frac{i(d + m) + d_i + 1}{d + i - 1} \right\}.$$

To prove the corollary, by the Lemma 2.1, we need to show that

$$\max_{2 \leq i \leq 5} \left\{ \frac{i(d + m) + d_i + 1}{d + i - 1} \right\} < \max_{1 \leq i \leq d'-1} \left\{ \frac{i(d + d' + m - i) + 1}{(d + i)} \right\}.$$

Let us take $i = \frac{d'}{2}$ on the right hand side of the above equation. Then

$$\max_{1 \leq i \leq d'-1} \left\{ \frac{i(d + d' + m - i) + 1}{(d + i)} \right\} \geq \frac{\left(\frac{d'}{2}\right) \left(d + d' + m - \frac{d'}{2}\right) + 1}{d + \frac{d'}{2}} = \frac{\frac{d'}{2} \left(d + m + \frac{d'}{2}\right) + 1}{d + \frac{d'}{2}} \geq \frac{d'}{2}$$

and we have a simple bound for the quantity in question from below. Using this bound, a case analysis for any $i = 1, \dots, 5$, shows that

$$\frac{i(d + m) + d_i + 1}{d + i - 1} \leq \frac{d'}{2}$$

since $d \geq 6$. This finishes the case $4d(d + m) \leq d'^2$ and hence the proof.

3 The functional equation

In this section we comment on the Corollary 1.15, namely that, the local normal zeta function of a group G as in Theorem 1.7, satisfies a functional equation of the form

$$\zeta_{G,p}^{\triangleleft}(s)|_{p \mapsto p^{-1}} = (-1)^{d+d'+m} p^{\binom{d+d'+m}{2} - (2d+d'+m)s} \cdot \zeta_{G,p}^{\triangleleft}(s).$$

By the Observation 2 on page 1013 [21], in our formulation, this is equivalent to

$$A(p, p^{-s})|_{p \mapsto p^{-1}} = (-1)^{d'-1} p^{\binom{d'}{2}} \cdot A(p, p^{-s}),$$

where

$$A(p, p^{-s}) = W_0(p, p^{-s}) + \sum_{i=1}^k \sum_{\mathbb{B}_i} \mathbf{n}_{\mathbb{B}_i}(p) W_{\mathbb{B}_i}(p, p^{-s}).$$

Recall from Theorem 1.7 that

$$W_{\mathbb{B}_i}(\mathbf{X}, \mathbf{Y}) = I_{d'-i-1}(X_{d'-1}, \dots, X_{i+1}) E_{\mathbb{B}_i}(X_i, Y_{\mathbb{B}_i}) I_{i-1}(Y_{\mathbb{B}_{i-1}}, \dots, Y_{\mathbb{B}_1}).$$

To prove the Corollary 1.15 it is enough to show that each of the summands $\mathbf{n}_{\mathbb{B}_i}(p) W_{\mathbb{B}_i}(\mathbf{X}, \mathbf{Y})$, for $i \geq 0$, satisfy the same functional equation as $A(p, p^{-s})$. We shall first establish inversion properties for the $W_{\mathbb{B}_i}(\mathbf{X}, \mathbf{Y})$, and secondly define what the formal inversion of $p \mapsto p^{-1}$ means for the coefficients $\mathbf{n}_{\mathbb{B}_i}(p)$.

Following Igusa [15], it is proved in [22, Theorem 4] that for $\mathbf{U} = (U_1, \dots, U_n) = (p^{-a_1 s + b_1}, \dots, p^{-a_n s + b_n})$ we have

$$I_n(\mathbf{U})|_{U_i \mapsto U_i^{-1}} = (-1)^n p^{\binom{n+1}{2}} I_n(\mathbf{U}),$$

where $I_n(\mathbf{U})$ as in Definition 1.5. Here the map $U_i \mapsto U_i^{-1}$ corresponds to $p \mapsto p^{-1}$.

So for each

$$W_{\mathbb{B}_i}(\mathbf{X}, \mathbf{Y}) = I_{d'-i-1}(X_{d'-1}, \dots, X_{i+1}) E_{\mathbb{B}_i}(X_i, Y_{\mathbb{B}_i}) I_{i-1}(Y_{\mathbb{B}_{i-1}}, \dots, Y_{\mathbb{B}_1})$$

we get functional equation coefficients $(-1)^{d'-i-1} p^{\binom{d'-i}{2}}$ and $(-1)^{i-1} p^{\binom{i}{2}}$ from the Igusa factors. It remains to determine the functional equation for $E_{\mathbb{B}_i}$. This is now easy, since we have the explicit formula

$$E_{\mathbb{B}_i}(X_i, Y_{\mathbb{B}_i}) = \frac{p^{-d_{\mathbb{B}_i}} Y_{\mathbb{B}_i} - p^{-n_i} X_i}{(1 - X_i)(1 - Y_{\mathbb{B}_i})},$$

which yields

$$E_{\mathbb{B}_i}(X_i, Y_{\mathbb{B}_i})|_{p \mapsto p^{-1}} = p^{n_i + d_{\mathbb{B}_i}} E_{\mathbb{B}_i}(X_i, Y_{\mathbb{B}_i}).$$

Here $n_i = i(d' - i)$ is the dimension of the Grassmannian $\mathbb{G}(n - 1, d' - 1)$, and $d_{\mathbb{B}_i}$ is the dimension of the component $F_{\mathbb{B}_i}$ on $F_{i-1}(\mathfrak{P}_G)$.

Recall, that $\mathbf{n}_{\mathbb{B}_i}(p)$ denoted the number of \mathbb{F}_p -points on certain varieties. We shall formally define $\mathbf{n}_{\mathbb{B}_i}(p)_{p \mapsto p^{-1}} = p^{-d_{\mathbb{B}_i}} \mathbf{n}_{\mathbb{B}_i}(p)$. This can be thought of coming from the fact that the Hasse-Weil zeta function has a functional equation if the variety is smooth and absolutely irreducible. In that setting, together the rationality and the Riemann hypothesis for the Hasse-Weil zeta function imply that if the components of the Fano varieties are smooth and absolutely irreducible then

$$\mathbf{n}_{\mathbb{B}_i}(p) = p^{d_{\mathbb{B}_i}} + \cdots + 1 + (-1)^m \sum_j \pi_j,$$

where $\pi_j \in \mathbb{C}$ and $|\pi_j| = \sqrt{p}$. Further, the functional equation of the zeta function implies that $\pi_j \mapsto \frac{p^{d_{\mathbb{B}_i}-1}}{\pi_j}$ induces a permutation of the set $\{\pi_j\}$. It follows that the inversion $\mathbf{n}_{\mathbb{B}_i}(p)|_{p \mapsto p^{-1}} = p^{-d_{\mathbb{B}_i}} \mathbf{n}_{\mathbb{B}_i}(p)$ is well-defined. This yields

$$W_{\mathbb{B}_i}(\mathbf{X}, \mathbf{Y})|_{p \mapsto p^{-1}} = (-1)^{d'-i-1+(i-1)} p^{\binom{d'-i}{2} + n_i + \binom{i}{2}} W_{\mathbb{B}_i}(\mathbf{X}, \mathbf{Y}).$$

Since, $\binom{d'-i}{2} + n_i + \binom{i}{2} = \binom{d'}{2}$, we get

$$W_{\mathbb{B}_i}(\mathbf{X}, \mathbf{Y})|_{p \mapsto p^{-1}} = (-1)^{d'} p^{\binom{d'}{2}} W_{\mathbb{B}_i}(\mathbf{X}, \mathbf{Y}),$$

as required.

A paper by Debarre and Manivel [6] shows that if \mathfrak{P}_G is a generic hypersurface, and moreover a complete intersection over an algebraically closed field, then the Fano variety is smooth, connected and has the expected dimension, i.e.,

$$d_{\mathbb{B}_i} = i(d' - i) - \binom{\frac{d'}{2} + (i - 1)}{i - 1}.$$

4 Lattices and points in the projective space

A lattice Λ' is an additive subgroup of $\mathbb{Z}_p^{d'}$. We say that Λ' is maximal in its homothety class if $\Lambda' \leq \mathbb{Z}_p^{d'}$ but $p^{-1}\Lambda' \not\leq \mathbb{Z}_p^{d'}$. The lattices which are maximal in their class are enumerated by elementary divisor types.

The *type* of

$$\Lambda' \cong \text{diag}(\underbrace{p^{r_{i_1} + \cdots + r_{i_l}}, \dots, p^{r_{i_1} + \cdots + r_{i_l}}}_{i_1}, \underbrace{p^{r_{i_2} + \cdots + r_{i_l}}, \dots, p^{r_{i_2} + \cdots + r_{i_l}}}_{i_2 - i_1}, \dots, \underbrace{1, \dots, 1}_{d' - i_l})$$

is denoted by $\nu = (I, r_I)$, where $I = \{i_1, \dots, i_l\}_< \subseteq \{1, \dots, d' - 1\}$, with $i_1 < i_2 < \cdots < i_l$, and the vector $r_I = (r_{i_1}, \dots, r_{i_l})$ records the values of the

r_{i_j} . If we are only interested in the indices appearing in the type, and not the exact values of the r_{i_j} , we say that Λ' has *flag type I*.

Let Λ' be a maximal lattice of type $\nu = (I, r_I)$, as above. The group $\mathrm{GL}_{d'}(\mathbb{Z}_p)$ acts transitively on the set of maximal lattices, and the Orbit-Stabiliser Theorem gives a 1-1 correspondence between

$$\{\text{maximal lattices of type } \nu\} \xrightarrow{1-1} \mathrm{GL}_{d'}(\mathbb{Z}_p)/G_\nu, \quad (4)$$

where G_ν is the stabiliser of the diagonal matrix

$$\mathrm{diag}(\underbrace{p^{r_{i_1}+\dots+r_{i_l}}, \dots, p^{r_{i_1}+\dots+r_{i_l}}}_{i_1}, \underbrace{p^{r_{i_2}+\dots+r_{i_l}}, \dots, p^{r_{i_2}+\dots+r_{i_l}}}_{i_2-i_1}, \dots, \underbrace{1, \dots, 1}_{d'-i_l})$$

in $\mathrm{GL}_{d'}(\mathbb{Z}_p)$.

The matrices in the stabiliser G_ν are of the form

$$\left(\begin{array}{c|c|c|c|c} \mathrm{GL}_{i_1}(\mathbb{Z}_p) & * & * & \dots & * \\ \hline p^{r_{i_1}}\mathbb{Z}_p & \mathrm{GL}_{i_2-i_1}(\mathbb{Z}_p) & * & \dots & * \\ \hline p^{r_{i_1}+r_{i_2}}\mathbb{Z}_p & p^{r_{i_2}}\mathbb{Z}_p & \mathrm{GL}_{i_3-i_2}(\mathbb{Z}_p) & \dots & * \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline p^{r_{i_1}+\dots+r_{i_{l-1}}}\mathbb{Z}_p & p^{r_{i_2}+\dots+r_{i_{l-1}}}\mathbb{Z}_p & p^{r_{i_3}+\dots+r_{i_{l-1}}}\mathbb{Z}_p & \dots & \mathrm{GL}_{d'-i_l}(\mathbb{Z}_p) \end{array} \right)$$

where $*$ indicates an arbitrary matrix.

By identifying Λ' with a coset representative βG_ν , we can think of the columns of β as points in $\mathbb{P}^{d'-1}(\mathbb{Z}_p)$ and then list them as $\beta_{i_j, k}$ where $i_j \in I$ indicates the block that $\beta_{i_j, k}$ belongs to, and $k \in \{i_{j-1} + 1, \dots, i_j\}$ is the running index across the columns of the matrix. Moreover, by the action of the stabiliser, we can multiply any column $\beta_{i_j, k}$ by a unit, add multiples of $\beta_{i_{j_1}, k_1}$ to $\beta_{i_{j_2}, k_2}$, whenever $k_2 > k_1$ (and necessarily $i_{j_2} \geq i_{j_1}$), and also add $p^{r_{i_{j_1}}+\dots+r_{i_{j_2}-1}}\beta_{i_{j_2}, k_2}$ to $\beta_{i_{j_1}, k_1}$ when $j_2 > j_1$. If $b_{i_{j_1}, n}^{i_{j_2}, m}$ denotes the (n, m) -entry of β , the above operations imply that $b_{i_{j_1}, n}^{i_{j_2}, m} \in \mathbb{Z}_p/(p^{r_{i_{j_1}}+\dots+r_{i_{j_2}-1}})$. This observation enables us to compute the number of lattices of a fixed elementary divisor type.

Definition 4.1. For each lattice Λ' of type (I, r_I) we define the *multiplicity* of Λ' , which we denote by $\mu(\Lambda')$, to be the number of lattices of fixed type (I, r_I) , divided by the factor $b_I(p)$ which counts the number of \mathbb{F}_p -points on the corresponding flag variety.

We now define an expression which encodes this multiplicity as a function of (I, r_I) . One can check that in order to compute $\mu(\Lambda')$ we may assume that $b_{i_{j_1}, n}^{i_{j_2}, m} \in p\mathbb{Z}_p/(p^{r_{i_{j_1}}+\dots+r_{i_{j_2}-1}})$. The function $\mu : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ will measure the size of the set of $x \in p\mathbb{Z}_p/(p^a)$ of a fixed p -adic valuation as follows:

Definition 4.2. Let a, b be fixed positive integers. We define a binary function $\mu : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ as follows

$$\mu(a, b) := |\{x \in p\mathbb{Z}_p/(p^a) : v_p(x) = b\}| = \begin{cases} 1 & \text{if } a = b \\ p^{a-b}(1 - p^{-1}) & \text{if } a > b \\ 0 & \text{otherwise.} \end{cases}$$

This definition extends naturally to a $(n+1)$ -ary function $\mu : \mathbb{N} \times \mathbb{N}^n \longrightarrow \mathbb{N}$; if $\mathbf{b} = (b_1, \dots, b_n)$ is a vector then

$$\mu(a; \mathbf{b}) := |\{\mathbf{x} \in (p\mathbb{Z}_p/(p^a))^n : v_p(x_i) = b_i\}|.$$

We note that $\mu(a; b_1, b_2, b_3, \dots, b_n) = \mu(a, b_1)\mu(a, b_2) \dots \mu(a, b_n)$ and

$$\sum_{b=1}^a \mu(a, b) = p^{a-1}. \quad (5)$$

The next result records the multiplicity of a lattice of given type $\nu = (I, r_I)$.

Proposition 4.3. Let Λ' be a maximal lattice of type $\nu = (I, r_I)$ with a coset representative βG_ν under the 1-1 correspondence in (4), where $I = \{i_1, \dots, i_l, i_{l+1}\}$ and $\beta \in \text{GL}_{d'}(\mathbb{Z}_p)$. Write $b_{i_{j_1}, k}^{i_{j_2}, m}$ for the (k, m) entry of β , where $k \in \{i_{j_1-1} + 1, \dots, i_{j_1}\}$ and $m \in \{i_{j_2-1} + 1, \dots, i_{j_2}\}$, so that the pair (i_{j_1}, i_{j_2}) indicates the block of this entry. Then

$$\begin{aligned} \mu(\Lambda') &= \prod_{\substack{j_1, j_2 \in \{1, \dots, l, l+1\} \\ j_1 < j_2}} \prod_k \prod_m \sum_{\substack{r_{i_{j_1}} + \dots + r_{i_{j_2-1}} \\ b_{i_{j_1}, k}^{i_{j_2}, m} = 1}} \mu(r_{i_{j_1}} + \dots + r_{i_{j_2-1}}, b_{i_{j_1}, k}^{i_{j_2}, m}) \\ &= \prod_{\substack{j_1, j_2 \in \{1, \dots, l, l+1\} \\ j_1 < j_2}} p^{i_{j_1} i_{j_2} (r_{i_{j_1}} + \dots + r_{i_{j_2-1}} - 1)} = p^{-\dim \mathcal{F}_I + \sum_{i_j \in I} (d' - i_j) i_j r_{i_j}}, \end{aligned}$$

where $d' = i_{l+1}$ and $I = \{i_1, \dots, i_l, i_{l+1}\}$.

Proof. By excluding the factor $b_I(p)$, we can assume that we are inside the first congruence subgroup, see [18] page 327 for details. Now the result follows by carefully enumerating the entries in the matrix β and by applying the equation (5) above. \square

5 Grassmannians and flag varieties

The set of $(k-1)$ -planes in $\mathbb{P}^{d'-1} = \mathbb{P}(V)$ admits a variety structure via the standard Plücker embedding into $\mathbb{P}^N = \mathbb{P}(\bigwedge^k V)$, where $N = \binom{d'}{k}$.

We can give local affine coordinates for open subsets on the Grassmannian $G(k, d')$ in the following way. Let $\Gamma \subset V$ be a subspace of dimension $d' - k$ corresponding to a multivector $\omega \in \bigwedge^{d'-k} V$. Now ω can be thought of as a homogeneous linear form on $\mathbb{P}(\bigwedge^k V)$, and we define $U \subset \mathbb{P}(\bigwedge^k V)$ to be the affine open subset defined by $\omega \neq 0$. Thus $U \cap G(k, V)$ is just the set of k -dimensional subspaces $\Delta \subset V$ that are complementary to Γ .

To give this construction in coordinates, let $\Gamma \subset V$ be a subspace of dimension $d' - k$, spanned by $e_{k+1}, \dots, e_{d'} \in \mathbb{F}_p^{d'}$, say. Then $U \cap G(k, V)$ is the subset of spaces Δ such that the $k \times d'$ matrix M_Δ whose first $k \times k$ minor is non-zero. It follows that any $\Delta \in U \cap G(k, V)$ can be represented as the row space of a unique matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & a_{11} & a_{12} & \dots & a_{1,d'-k} \\ 0 & 1 & 0 & \dots & 0 & a_{21} & a_{22} & \dots & a_{2,d'-k} \\ 0 & 0 & 1 & \dots & 0 & a_{31} & a_{32} & \dots & a_{3,d'-k} \\ \vdots & & & & & & & & \\ 0 & 0 & 0 & \dots & 1 & a_{k1} & a_{k2} & \dots & a_{k,d'-k} \end{pmatrix} \quad (6)$$

and vice versa. The entries of this matrix then give the bijection of $U \cap G(k, V)$ with $\mathbb{F}_p^{k(d'-k)}$.

A flag variety \mathcal{F}_I is a generalisation of a Grassmannian. Here $I = \{i_1, \dots, i_l\}_< \subseteq \{1, \dots, d'\}$ and \mathcal{F}_I is a subvariety of a product of Grassmannians defined by the incidence correspondence

$$\mathcal{F}_I := \{(\Pi_1, \dots, \Pi_l) : \Pi_1 \subset \dots \subset \Pi_l\} \subset \mathbb{G}(i_1-1, d'-1) \times \dots \times \mathbb{G}(i_l-1, d'-1).$$

By generalising the above construction for affine coordinates of points on the Grassmannian, we can choose local coordinates for points on the flag variety.

We can also define flags of type $I = \{i_1, \dots, i_l\}$ starting from the lattices Λ' of flag type I in the following way. As before, let Λ' be a lattice of type $\nu = (I, r_I)$ which corresponds to the coset βG_ν under the 1-1 correspondence (4), for some fixed $\beta \in \text{GL}_{d'}(\mathbb{Z}_p)$. Write $\beta_{i_j,k}$ for the columns of β . Let $\overline{\beta_{i_j,k}}$ denote the reduction of $\beta_{i_j,k} \bmod p$. These $\overline{\beta_{i_j,k}}$ can be thought of as points in $\mathbb{P}^{d'-1}(\mathbb{F}_p)$. We define vector spaces for each $i_j \in I$ by setting

$$V_{i_j} = \langle \overline{\beta_{i_1,1}}, \dots, \overline{\beta_{i_j,i_j}} \rangle_{\mathbb{F}_p} < \mathbb{F}_p^{d'}.$$

We observe that $\dim_{\mathbb{F}_p}(V_{i_j}) = i_j$. We call $(V_{i_j})_{i_j \in I}$ the flag of type I associated to Λ' . Obviously, a number of lattices will give the same flag mod p .

Definition 5.1. Let $\mathfrak{F} \in \mathcal{F}_I$, where $I = \{i_1, \dots, i_l\}_<$, and let Λ' be a maximal lattice of type I . Then Λ' is said to be a lift of \mathfrak{F} if its associated flag $(V_i)_{i \in I}$ is equal to \mathfrak{F} .

The stabiliser of each flag $\mathfrak{F} \in \mathcal{F}_I$ is the standard parabolic subgroup P_I in $\mathrm{GL}_{d'}(\mathbb{F}_p)$. Each of these parabolic subgroups contain the standard Borel subgroup of upper triangular matrices.

Definition 5.2. Two maximal lattices Λ'_1 and Λ'_2 of type (I, r_I) are said to be *equivalent* if they are lifts of the same flag $\mathfrak{F} \in \mathcal{F}_I$.

This defines an equivalence relation \sim on the set of maximal lattices. We note that $\Lambda'_1 \sim \Lambda'_2$ if and only if the respective coset representatives of Λ'_1 and Λ'_2 differ by an action of right multiplication by the parabolic subgroup P_I in $\mathrm{GL}_{d'}(\mathbb{Z}_p)$. In particular, this observation allows us to change the coordinates of the flag, as well as providing a uniform way to move between different cosets corresponding to liftings of the same flag.

Let Λ' be a maximal lattice of type $I = \{i_1, \dots, i_l\}_<$, corresponding to β with columns $\{\beta_{i_1,1}, \dots, \beta_{i_1,i_1}, \beta_{i_2,i_1+1}, \dots, \beta_{i_2,i_2}, \dots, \beta_{i_l,i_l}, \dots, \beta_{d',d'}\}$, and entries $b_{i_j,k}^{i_{j2},m}$. We will construct a lattice Λ'' such that $\Lambda' \sim \Lambda''$. To do this, first set

$$B = \begin{pmatrix} \lambda_1 & \lambda_1 & \lambda_1 & \dots & \lambda_1 \\ & \lambda_2 & \lambda_2 & \dots & \lambda_2 \\ & & \lambda_3 & \dots & \lambda_3 \\ & & & \ddots & \vdots \\ & & & & \lambda_{d'} \end{pmatrix}, \quad (7)$$

where all the entries are p -adic units. This is clearly an element of the standard Borel subgroup, so B is contained in each of the parabolic subgroups P_I , and thus multiplying βG_ν from the right by B leaves the associated flag invariant. We have now moved to the coset $\beta G_\nu B = \alpha G_\nu$, and as before we shall denote the columns of α by $\{\alpha_{i_1,1}, \dots, \alpha_{i_1,i_1}, \alpha_{i_2,i_1+1}, \dots, \alpha_{i_2,i_2}, \dots, \alpha_{i_l,i_l}, \dots, \alpha_{d',d'}\}$, which expand as

$$\alpha_{i_j,k} = \begin{pmatrix} a_{i_j,k}^{i_{l+1},d'} \\ \vdots \\ a_{i_j,k}^{i_{j2},m} \\ \vdots \\ a_{i_j,k}^{i_1,1} \end{pmatrix} = \lambda_1 \begin{pmatrix} b_{i_1,1}^{i_{l+1},d'} \\ \vdots \\ b_{i_1,1}^{i_{j2},m} \\ \vdots \\ b_{i_1,1}^{i_1,1} \end{pmatrix} + \dots + \lambda_n \begin{pmatrix} b_{i_{j1},n}^{i_{l+1},d'} \\ \vdots \\ b_{i_{j1},n}^{i_{j2},m} \\ \vdots \\ b_{i_{j1},n}^{i_1,1} \end{pmatrix} + \dots + \lambda_k \begin{pmatrix} b_{i_{j1},k}^{i_{l+1},d'} \\ \vdots \\ b_{i_{j1},k}^{i_{j2},m} \\ \vdots \\ b_{i_{j1},k}^{i_1,1} \end{pmatrix}. \quad (8)$$

However, this is not enough, since even if the reductions mod p span the same flags, we will also require that the coordinates of the $\alpha_{i_j,k}$ lie in the same quotient ring as the coordinates of $\beta_{i_j,k}$. This is not the case here, since e.g. $a_{i_j,k}^{i_{l+1},d'} \in p\mathbb{Z}_p/(p^{r_{i_1}+\dots+r_{i_l}})$, while $b_{i_j,k}^{i_{l+1},d'} \in p\mathbb{Z}_p/(p^{r_{i_j}+\dots+r_{i_l}})$. In order to make the coordinates of the $\alpha_{i_j,k}$ to lie in the same quotient ring as those of $\beta_{i_j,k}$ we recall that the $\alpha_{i_j,k}$ are unique up to the action of right-multiplication by G_ν , and this will give us a reduction of the form $b_{i_{j1},n}^{i_{j2},m}$

mod $p^{r_{i_1} + \dots + r_{i_j} - 1}$, so we may indeed assume each of the coordinates of the $\alpha_{i_j, k}$ lie in the same quotient of $p\mathbb{Z}_p$ as the coordinates of $\beta_{i_j, k}$. To avoid unnecessary notation, all the coordinates where this kind of reduction has been performed will be denoted by $\hat{a}_{i_j, k}^{i_{j_2}, m}$.

6 Solution sets for systems of linear congruences

Let Λ' be a maximal lattice of type $I = \{i_1, \dots, i_l\}_<$ corresponding to the coset βG_ν . Let $\{\beta_{i_1, 1}, \dots, \beta_{i_1, i_1}, \beta_{i_2, i_1+1}, \dots, \beta_{i_2, i_2}, \dots, \beta_{i_l, i_l}\}$ denote the columns of β . We want to determine the index of the kernel of the following system of linear congruences for each $i_j \in I$ and $k \in \{i_{j-1} + 1, \dots, i_j\}$

$$\mathbf{g}M(\beta_{i_j, k}) \equiv 0 \pmod{p^{r_{i_j} + \dots + r_{i_l}}}, \quad (9)$$

where $\mathbf{g} = (g_1, \dots, g_d) \in \mathbb{Z}_p^d$. And $M(\mathbf{y})$ is the matrix of relations in a presentation of Γ as in Definition 1.1.

We shall denote the index of the kernel of the above system of linear congruences by $w'(\Lambda')$ and call it the weight function of the lattice Λ' . This index has a group theoretical significance, which we shall explain in Section 8.

The main point of this section is determine the value of this function for different kinds of lattices Λ' .

It is clear from (9) that the solution set of this system of linear congruences depends on whether the matrix $M(\beta_{i_j, k})$ has full rank or not, i.e., depending on whether the determinant of the matrix $M(\mathbf{y})$ evaluated at $\beta_{i_j, k}$ is zero mod p or not. Recall, that we insist that the Pfaffian is not identically zero. The vanishing locus of the square root of the determinant of $M(\mathbf{y})$ defines a hypersurface in $\mathbb{P}^{d'-1}(\mathbb{F}_p)$. This hypersurface is called the Pfaffian hypersurface, as in Definition 1.1, and we will denote it by \mathfrak{P}_G .

Let us briefly sketch the connection with the arithmetic geometry which arises in this context and the type of geometric problems we encounter. For convenience, let us consider a lattice of type $(p^{r_{i_1}}, \dots, p^{r_{i_1}}, 1, \dots, 1)$ which gives rise to column vectors $\{\beta_1, \dots, \beta_{i_1}, \beta_{i_1+1}, \dots, \beta_{d'}\}$ so that $\beta_{i_1+1}, \dots, \beta_{d'}$ give redundant congruence conditions and there exists $k \in \{1, \dots, i_1\}$ such that $\det^{\frac{1}{2}} M(\beta_k) \equiv 0 \pmod{p}$ with respect to the system of linear congruences.

The $\beta_1, \dots, \beta_{i_1}$ are unique only up to multiplication by a unit and addition of \mathbb{Z}_p -multiple of β_i to β_j for any $i, j \in \{1, \dots, i_1\}$, so it follows that either all mod p reductions $\overline{\beta_k}$ of $\beta_k \in \mathbb{P}^{d'-1}(\mathbb{Z}_p)$ are $\overline{\beta_1}, \dots, \overline{\beta_{i_1}} \in \mathfrak{P}_G$ or all $\overline{\beta_1}, \dots, \overline{\beta_{i_1}} \notin \mathfrak{P}_G$. These cases correspond to whether the $(i_1 - 1)$ -plane $\langle \overline{\beta_1}, \dots, \overline{\beta_{i_1}} \rangle$ is contained on \mathfrak{P}_G or not.

As mentioned in Section 5 the $(i_1 - 1)$ -planes in $\mathbb{P}^{d'-1}$ are parametrised by a Grassmannian variety. The $(i_1 - 1)$ -planes that are contained on \mathfrak{P}_G

are parametrised by the Fano variety $F_{i_1-1}(\mathfrak{P}_G)$. This Fano variety may be reducible, and we will denote its components by $F_{\mathfrak{B}_{i_1}}$.

The most convenient description of this Fano variety is as a subvariety of the Grassmannian. Let us now describe a construction of its defining ideals on the Grassmannian. Choose an $(i_1 - 1)$ -plane on the Pfaffian. Using the construction of the affine coordinates given in Section 5, we can identify an open neighbourhood of this point on the Grassmannian $\mathbb{G}(i_1 - 1, d' - 1)$ as an affine space. In this neighbourhood, a hyperplane can be represented as the row span of the matrix (6) as in Section 5, with coefficients μ_i , say. Restricting this hyperplane onto the Pfaffian will give us a bihomogeneous polynomial of bidegree $(\frac{d}{2}, \frac{d}{2})$. The coefficients of the monomials in the μ_i will give us the defining ideals of the Fano variety on the Grassmannian. We shall denote the dimension of the Fano variety by $d_{\mathfrak{B}_{i_1}}$ and its codimension on the Grassmannian by $c_{\mathfrak{B}_{i_1}}$.

It is however, not enough to know that the determinant of the matrix of relations is zero mod p at the smooth points. We know the exact rank of the matrix or relations from the following lemma.

Lemma 6.1 (Voll [20]). *If $\mathfrak{P}_G \subseteq \mathbb{P}^{d'-1}(\mathbb{Q})$ is smooth, then the rank of the matrix of relations G at any smooth point is equal to $d - 2$.*

In particular, the matrix of relations at a smooth point always contains a $(d - 2) \times (d - 2)$ minor with a p -adic unit determinant.

Definition 6.2. Let $A \in \text{Mat}_{d,l}(\mathbb{Z}_p)$. Then the corank of A is defined to be $r = d - n$, where $n \times n$ is the largest p -adic unit minor.

In view of, (9) we are interested only in the index of the kernel of the set of equations

$$\mathbf{g}M(\beta_{i_j,k}) \equiv 0 \pmod{p^{r_{i_j} + \dots + r_{i_l}}}$$

for $i_j \in I = \{i_1, \dots, i_l\}$ and $k \in \{i_{j-1} + 1, \dots, i_j\}$.

Remark 6.3. It is important to note that only the index matters; this is the single fact that makes calculating normal zeta functions of class two nilpotent groups so much easier than calculating the subgroup zeta function.

The information we need from the matrices $M(\beta_{i_j,k})$ for the computation of the index is independent of the basis chosen. In the next few pages we show that, by a suitable choice of homogeneous coordinates, we can arrange the points on the Pfaffian so that the solution sets of these congruences form a chain.

Let us first make a reduction.

Lemma 6.4. *Let Λ' be a lattice of flag type $I = J_1 \cup J_2$ where $J_1 = \{i_1, \dots, i_l\}$ and $J_2 = \{i_{l+1}, \dots, i_m\}$. Suppose that the intersection of the flag $(V_{i_j})_{i_j \in I}$ associated with Λ' with the Pfaffian hypersurface is the flag $\mathfrak{F} \in \mathcal{F}_{J_1}$. Then*

$$w'(\Lambda') = w'(\Lambda', I, r_I) = w'(\Lambda', J_1, r_{J_1}) + w'(\Lambda', J_2, r_{J_2}),$$

where $J_1 = \{i_1, \dots, i_l\}$ and $J_2 = \{i_{l+1}, \dots, i_m\}$. Moreover,

$$w'(\Lambda', J_2, r_{J_2}) = d(r_{i_{l+1}} + \dots + r_{i_m}).$$

Proof. Let $\{\beta_{i_1,1}, \dots, \beta_{i_1,i_1}, \beta_{i_2,i_1+1}, \dots, \beta_{i_2,i_2}, \dots, \beta_{i_m,i_m}\}$ be the columns of β in βG_ν corresponding to Λ' . Then $M(\beta_{i_1,1}), \dots, M(\beta_{i_1,i_1}), M(\beta_{i_2,i_1+1}), \dots, M(\beta_{i_l,i_l})$ have determinants zero mod p , and we can assume that the determinants of $M(\beta_{i_{l+1},i_{l+1}}), \dots, M(\beta_{i_m,i_m})$ are p -adic units. As before, we can do this since the $\beta_{i_j,k}$ are unique only up to the action of the stabiliser.

Writing this explicitly, we need to solve the following systems of linear congruences

$$\begin{aligned} \mathbf{g}M(\beta_{i_j,k}) &\equiv 0 \pmod{p^{r_{i_j} + \dots + r_{i_m}}} & \forall i_j \in J_1, k \in \{i_{j-1} + 1, \dots, i_j\}, \\ \mathbf{g} &\equiv 0 \pmod{p^{r_{i_j} + \dots + r_{i_m}}} & \forall i_j \in J_2. \end{aligned}$$

which is equivalent to the two independent sets of equations

$$\begin{aligned} \mathbf{g}M(\beta_{i_j,k}) &\equiv 0 \pmod{p^{r_{i_j} + \dots + r_{i_l}}} & \forall i_j \in J_1, k \in \{i_{j-1} + 1, \dots, i_j\}, \\ \mathbf{g} &\equiv 0 \pmod{p^{r_{i_{l+1}} + \dots + r_{i_m}}}. \end{aligned}$$

The first of these is equivalent to the congruence conditions that arise for Λ' of type (J_1, r_{J_1}) , while the second corresponds to the congruences conditions for $w'(\Lambda', J_2, r_{J_2})$. The latter is also equal to

$$w'(\Lambda', J_2, r_{J_2}) = d(r_{i_{l+1}} + \dots + r_{i_m})$$

as claimed. \square

Let us now consider the possible intersections of a flag $\mathfrak{F} \in \mathcal{F}_{J_1}$ of type $J_1 = \{i_1, \dots, i_l\}$ associated to Λ' with the Pfaffian hypersurface of G .

Case I: The intersection is empty

In the light of the previous lemma, first suppose that is $J_1 = \emptyset$. Then the weight function is

$$w'(\Lambda') = d\left(\sum_{i \in J_2} r_i\right) = d\left(\sum_{i \in I} r_i\right),$$

since $I = J_2$.

Case II: The intersection is a hyperplane

Next let us assume the intersection is a hyperplane.

If $J_1 = \{1\}$ then the desired result follows from the following:

Lemma 6.5 (Voll [22]). *Let $\xi \in \mathfrak{P}_G$, and let Λ' be a lattice of type $J_1 = \{1\}$ such that the flag associated with Λ' intersects the Pfaffian exactly at this fixed point. Then is $w'(\Lambda', 1, r_1) = dr_1 - 2 \min\{r_1, v_p(\det^{\frac{1}{2}} M(\beta_{1,1}))\}$.*

We include his proof here, since this is the template we shall use later on for a more complicated case analysis.

Proof. Choose $\beta_1 \in \mathbb{P}^{d'-1}(\mathbb{Z}_p)$ such that $\det^{\frac{1}{2}} M(\beta_{1,1}) \equiv 0 \pmod{p}$, and $\overline{\beta_{1,1}} = \xi \in \mathfrak{P}_G$. Let $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then we can choose local coordinates such that around any of the $\mathbf{n}_1(p)$ points of the Pfaffian hypersurface mod p the congruence conditions look like

$$\mathbf{g} \left(\begin{pmatrix} 0 & \det^{\frac{1}{2}} M(\beta_{1,1}) \\ -\det^{\frac{1}{2}} M(\beta_{1,1}) & 0 \end{pmatrix}, J, \dots, J \right) \equiv 0 \pmod{p^{r_1}}.$$

It follows that $w'(\Lambda', 1, r_1) = dr_1 - 2 \min\{r_1, v_p(\det^{\frac{1}{2}} M(\beta_1))\}$, as claimed. \square

Let Π be a (i_1-1) -plane on the Pfaffian, and Λ' a lattice of type $J_1 = \{i_1\}$, for $i_1 > 1$, such that Π is the intersection of the Pfaffian with the flag associated to Λ' , i.e. $\langle \overline{\beta_{i_1,1}}, \dots, \overline{\beta_{i_1,i_1}} \rangle = \Pi$

As in Section 5, we can consider the columns of α where $\alpha G_\nu = \beta G_\nu B$ instead β and so we are required to compute the index of the kernel of

$$\mathbf{g} M(\alpha_{i_1,k}) \equiv 0 \pmod{p^{r_{i_1}}},$$

for $k = 1, \dots, i_1$. We can conveniently write this as the augmented matrix

$$\mathbf{g} (M(\alpha_{i_1,1}) | M(\alpha_{i_1,2}) | \dots | M(\alpha_{i_1,i_1})) \equiv 0 \pmod{p^{r_{i_1}}}. \quad (10)$$

We need to consider separately all the possible components of the Fano variety $F_{i_1-1}(\mathfrak{P}_G)$. The first thing to consider is the rank of the system of congruences, which is determined by the number of linearly independent column vectors of the augmented matrix. By Lemma 6.1, the rank of the augmented matrix is always d , $d-1$ or $d-2$, so that corank is either 0, 1 or 2, respectively.

In fact, the corank is 2 only if the intersection is a single point. The next lemma demonstrates that the corank of the augmented matrix is either 0 or 1 for lines and higher dimensional subspaces.

Lemma 6.6. *Let $\langle \overline{\alpha_1}, \dots, \overline{\alpha_{i_1}} \rangle \in \mathfrak{P}_G$, for $i_1 \geq 2$, then set of congruences*

$$\mathbf{g}(M(\alpha_{i_1,1})|M(\alpha_{i_1,2})|\dots|M(\alpha_{i_1,i_1})) \equiv 0 \pmod{p^{r_{i_1}}}$$

has corank 1 or 0.

Proof. It is enough to prove the lemma for the case of lines, that is $i_1 = 2$, since the rank of the augmented matrix in (10) is minimal in the case of lines for general $i_1 \geq 2$. From Lemma 6.5 we know that around a smooth point the corank is 2. Choose homogeneous coordinates for this point (as a vector), and choose coordinates for another smooth point on the Pfaffian such that these two points span a line on the Pfaffian. These vectors are linearly independent and we can add multiples of one to the other and we can also multiply them by units. Thus without loss of generality (up to a permutation of entries) our vectors look like $\sigma = (\sigma_1, \dots, \sigma_{d-2}, 0, 1)$ and $\tau = (\tau_1, \dots, \tau_{d-2}, 1, 0)$, where $\sigma_i, \tau_i \in p\mathbb{Z}_p/(p^{r_2})$. We may assume that we are not in the diagonal case. Indeed, if the matrix of relations is a permutation of a diagonal matrix, then G is a direct product of Heisenberg groups, in which case the Pfaffian is not smooth, so we can exclude this case. Then the corank of $M(\sigma)$ is 2 and we only need to show that at least one column of $M(\tau)$ is linearly independent from the columns of $M(\sigma)$. But this is indeed the case, since the matrix of relations is anti-symmetric, so the unit entry in τ will necessarily be in a different row to any unit entry in $M(\sigma)$. Thus the corank is at most 1, and if we happen to have two more linearly independent vectors then the corank is 0. \square

First let us assume the corank is 0. Here the augmented matrix contains a p -adic unit minor of size $d \times d$, and so the desired result follows, since the congruences will reduce to

$$\mathbf{g} \equiv 0 \pmod{p^{r_{i_1}}}$$

as in Case I: The intersection is empty, hence $w'(\Lambda', i_1, r_{i_1}) = w'(i_1, r_{i_1}) = dr_{i_1}$.

Remark 6.7. It is for these components $F_{\mathfrak{B}_i}(\mathfrak{P}_G)$ for which we will set $\delta_{\mathfrak{B}_i} = 0$, as in statement of Theorem 1.7.

Finally, let us assume we are in the corank 1 case. As in the proof of Lemma 6.5, we can use row and column operations to get each of the $M(\alpha_{i_1,k})$ for $k \in \{1, \dots, i_1\}$ into the form

$$\begin{pmatrix} 0 & \det^{\frac{1}{2}} M(\alpha_{i_1,k}) \\ -\det^{\frac{1}{2}} M(\alpha_{i_1,k}) & 0 \end{pmatrix}, J, \dots, J,$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

In order to determine the index of \mathbf{g} we need to know the rank, the determinant and the elementary divisor type of the matrices $M(\alpha_{i_j,k})$, and these do not depend on the particular basis chosen. So it suffices to solve

$$\begin{aligned} \mathbf{g} \begin{pmatrix} 0 & \det^{\frac{1}{2}} M(\alpha_{i_1,1}) \\ -\det^{\frac{1}{2}} M(\alpha_{i_1,1}) & 0 \end{pmatrix}, J, \dots, J \Big) &\equiv 0 \pmod{p^{r_{i_1}}} \\ \mathbf{g} \begin{pmatrix} 0 & \det^{\frac{1}{2}} M(\alpha_{i_1,2}) \\ -\det^{\frac{1}{2}} M(\alpha_{i_1,2}) & 0 \end{pmatrix}, J, \dots, J \Big) &\equiv 0 \pmod{p^{r_{i_1}}} \\ &\vdots \\ \mathbf{g} \begin{pmatrix} 0 & \det^{\frac{1}{2}} M(\alpha_{i_1,i_1}) \\ -\det^{\frac{1}{2}} M(\alpha_{i_1,i_1}) & 0 \end{pmatrix}, J, \dots, J \Big) &\equiv 0 \pmod{p^{r_{i_1}}} \end{aligned}$$

simultaneously, and we read off

$$w'(\Lambda', i_1, r_{i_1}) = dr_{i_1} - \min\{r_{i_1}, v_p(\det^{\frac{1}{2}}(M(\alpha_{i_1,1}))), v_p(\det^{\frac{1}{2}}(M(\alpha_{i_1,2}))), \dots, v_p(\det^{\frac{1}{2}}(M(\alpha_{i_1,i_1})))\}. \quad (11)$$

Since we are assuming the corank is 1, we do not have the coefficient two in the minimum-function that we had in Lemma 6.5.

The different components of the Fano variety may give different weight functions. This is either due to the rank condition, or the possibility that the components have different dimensions. We shall denote the \mathbb{F}_p -points on the component $F_{\mathbb{B}_{i_1}}$ by $\mathbf{n}_{\mathbb{B}_{i_1}}$. If a component has corank 0, we get the same weight function as the empty intersection case, and then we set $\delta_{\mathbb{B}_{i_1}} = 0$.

Case III: The general intersection

Here we assume that the intersection of the flag of type $I = \{i_1, \dots, i_l\}_<$ with the Pfaffian is not just a hyperplane but a flag with subspaces contained on the Pfaffian. For simplicity, let us exclude the corank 0 cases, since these again will give the same weight function as the case of an empty intersection. As before, we shall consider the congruence conditions separately for different lattices lifting fixed flags on the Pfaffian. Moreover, as in the case of a hyperplane intersection we consider the points $\alpha_{i_j,k}$ defined in Section 5 equation (8), instead of the $\beta_{i_j,k}$. In order to solve the equations against the same modulus, we consider the set of equations

$$p^{r_{i_1} + \dots + r_{i_{j-1}}} \mathbf{g} M(\alpha_{i_j,k}) \equiv 0 \pmod{p^{r_{i_1} + \dots + r_{i_{j-1}} + r_{i_j} + \dots + r_{i_l}}},$$

where $i_j \in J_1$ and $k \in \{i_{j-1} + 1, \dots, i_j\}$.

For each fixed i_j , we get an augmented matrix as in the hyperplane case:

$$p^{r_{i_1} + \dots + r_{i_{j-1}}} \mathbf{g}(M(\alpha_{i_j,i_{j-1}+1}) | \dots | M(\alpha_{i_j,i_j})) \equiv 0 \pmod{p^{r_{i_1} + \dots + r_{i_{j-1}} + r_{i_j} + \dots + r_{i_l}}}.$$

With a suitable change of basis, each component of the augmented matrix reduces to

$$p^{r_{i_1}+\dots+r_{i_{j-1}}}\mathbf{g}\left(\begin{pmatrix} 0 & \det^{\frac{1}{2}}M(\alpha_{i_j,k}) \\ -\det^{\frac{1}{2}}M(\alpha_{i_j,k}) & 0 \end{pmatrix}, J, \dots, J\right) \equiv 0 \pmod{p^{r_{i_1}+\dots+r_{i_j}+\dots+r_{i_l}}}. \quad (12)$$

Moreover, as we vary $i_j \in J_1$, we get the whole system of congruences consisting of augmented matrices of the same form as those which arose in the case $J_1 = \{i_1\}$. As before, the elementary row and column operations do not change the elementary divisor type of the matrices $M(\alpha_{i_j,k})$, so without loss of generality we may assume all of the matrices are of the form above (12). We can then read off the weight function:

$$\begin{aligned} w'(\Lambda', J_1, r_{J_1}) &= d(r_{i_1} + \dots + r_{i_l}) \\ &\quad - \min\{r_{i_1} + \dots + r_{i_l}, v_p(\det^{\frac{1}{2}}M(\alpha_{i_1,1})), \dots, v_p(\det^{\frac{1}{2}}M(\alpha_{i_1,i_1})), \\ &\quad r_{i_1} + v_p(\det^{\frac{1}{2}}M(\alpha_{i_2,i_1+1})), \dots, r_{i_1} + v_p(\det^{\frac{1}{2}}M(\alpha_{i_2,i_2})), \dots, \\ &\quad r_{i_1} + \dots + r_{i_{l-1}} + v_p(\det^{\frac{1}{2}}M(\alpha_{i_l,i_{l-1}+1})), \dots, r_{i_1} + \dots + r_{i_{l-1}} + v_p(\det^{\frac{1}{2}}M(\alpha_{i_l,i_l}))\}. \end{aligned}$$

However, if the flag contains the subspaces of points, i.e. if $J_1 = \{1, i_2, \dots, i_l\}$, then

$$\begin{aligned} w'(\Lambda', J_1, r_{J_1}) &= d(r_1 + \dots + r_{i_l}) - \min\{r_1, v_p(\det^{\frac{1}{2}}M(\alpha_{1,1}))\} \\ &\quad - \min\{r_1 + \dots + r_{i_l}, v_p(\det^{\frac{1}{2}}M(\alpha_{1,1})), r_1 + v_p(\det^{\frac{1}{2}}M(\alpha_{i_2,2})), \dots, r_1 + v_p(\det^{\frac{1}{2}}M(\alpha_{i_2,i_2})), \\ &\quad r_1 + r_{i_2} + v_p(\det^{\frac{1}{2}}M(\alpha_{i_3,i_2+1})), \dots, r_1 + r_{i_2} + v_p(\det^{\frac{1}{2}}M(\alpha_{i_3,i_3})), \dots, \\ &\quad r_1 + r_{i_2} + \dots + r_{i_{l-1}} + v_p(\det^{\frac{1}{2}}M(\alpha_{i_l,i_{l-1}+1})), \dots, r_1 + r_{i_2} + \dots + r_{i_{l-1}} + v_p(\det^{\frac{1}{2}}M(\alpha_{i_l,i_l}))\}. \end{aligned} \quad (13)$$

7 p -adic valuations of determinants

In order to get an explicit expression for the weight function $w'(\Lambda')$ we are left with the determination of $v_p(\det^{\frac{1}{2}}M(\alpha_{i_j,k}))$. This is easy from the following lemma:

Lemma 7.1. *Let $\alpha_{i_j,k}$ be the columns of α that we constructed in Section 5, equation (8), with the appropriate reductions. Then*

$$v_p(\det^{\frac{1}{2}}M(\alpha_{i_j,k})) = \min\{v_p(a_1), \dots, v_p(a_{c_{\mathbb{B}_k}})\},$$

where $c_{\mathbb{B}_k}$ is the codimension of the component $F_{\mathbb{B}_k}$ of $F_{k-1}(\mathfrak{P}_G)$ in $\mathbb{G}(k-1, d'-1)$.

Proof. First we note that we can expand $\det^{\frac{1}{2}}M(\alpha_{i_j,k})$ as a bihomogeneous polynomial of bidegree $(\frac{d}{2}, \frac{d}{2})$ in the entries of $\alpha_{i_j,k}$, and in the λ_n , where

λ_n are the p -adic unit entries of the matrix B (see equation (7)) which we used in order to move between lattices lifting the same flag. Writing this as a polynomial in the λ_n , the coefficients which appear are the defining ideals of the Fano variety on the Grassmannian. Furthermore, the p -adic valuation of any monomial $v_p(\lambda_1^{\varepsilon_1} \cdots \lambda_n^{\varepsilon_n}) = 1$, where $\sum_{i=1}^n \varepsilon_i = \frac{d}{2}$. Hence

$$v_p(\det^{\frac{1}{2}} M(\alpha_{i_j, k})) \geq \min\{v_p(a_1), \dots, v_p(a_{c_{\mathbb{B}_k}})\},$$

where a_i are polynomials in the entries of α . To see that equality holds, suppose for a contradiction that the left hand side is strictly bigger than the right hand side, so we have the congruence

$$\sum_{\varepsilon_1 + \dots + \varepsilon_n = \frac{d}{2}} \lambda_1^{\varepsilon_1} \cdots \lambda_n^{\varepsilon_n} a_{\varepsilon_1, \dots, \varepsilon_n} \equiv 0 \pmod{p^{\kappa+1}},$$

where $\kappa = \min\{v_p(a_1), \dots, v_p(a_{c_{\mathbb{B}_k}})\}$. Then

$$p^\kappa \left(\sum_{\varepsilon_1 + \dots + \varepsilon_n = \frac{d}{2}} \lambda_1^{\varepsilon_1} \cdots \lambda_n^{\varepsilon_n} a'_{\varepsilon_1, \dots, \varepsilon_n} \right) \equiv 0 \pmod{p^{\kappa+1}},$$

where $p^\kappa a'_{\varepsilon_1, \dots, \varepsilon_n} = a_{\varepsilon_1, \dots, \varepsilon_n}$ and thus

$$\sum_{\varepsilon_1 + \dots + \varepsilon_n = \frac{d}{2}} \lambda_1^{\varepsilon_1} \cdots \lambda_n^{\varepsilon_n} a'_{\varepsilon_1, \dots, \varepsilon_n} \equiv 0 \pmod{p}$$

for all possible p -adic units λ_n . It follows that

$$a'_{\varepsilon_1, \dots, \varepsilon_n} \equiv 0 \pmod{p}$$

for all possible choices of $\{\varepsilon_i\}$ with $\sum_{i=1}^n \varepsilon_i = \frac{d}{2}$. But this is a contradiction, since at least one of the $a'_{\varepsilon_1, \dots, \varepsilon_n}$ is a p -adic unit. \square

We have set up the new coordinates such that the set of valuations $v_p(\det^{\frac{1}{2}} M(\alpha_{i_j, k_1}))$ is contained in the set of valuations $v_p(\det^{\frac{1}{2}} M(\alpha_{i_j, k_2}))$ whenever $k_1 < k_2$. By applying Lemma 7.1, we can expand the determinants in the weight function.

We start with the case $I = \{i_1\}$ so the intersection of the flag variety with the Pfaffian hypersurface is a hyperplane. In view of, (11) we need to compute

$$\min\{r_{i_1}, v_p(\det^{\frac{1}{2}}(M(\alpha_{i_1, 1}))), v_p(\det^{\frac{1}{2}}(M(\alpha_{i_1, 2}))), \dots, v_p(\det^{\frac{1}{2}}(M(\alpha_{i_1, i_1})))\}. \quad (14)$$

From the definition of the $\alpha_{i_j, k}$, we get

$$v_p(\det^{\frac{1}{2}}(M(\alpha_{i_1, k}))) = \min\{v_p(a_1), v_p(a_2), \dots, v_p(a_{c_{\mathbb{B}_k}})\}$$

for each $k \in \{1, \dots, i_1\}$, where the a_n are polynomials in the coordinates of the points $\alpha_{i_1,k}$, and hence in the coordinates of the $\beta_{i_1,k}$. Substituting in (14) and cancelling the extra minima and any terms that appear more than twice, we obtain

$$w'(\Lambda', i_1, r_{i_1}) = dr_{i_1} - \min\{r_{i_1}, v_p(a_1), \dots, v_p(a_{c_{\beta_{i_1}}})\}.$$

Finally, we consider the case of a general lattice with weight function as in (13). Here we need to expand

$$\begin{aligned} & \min\{r_{i_1} + \dots + r_{i_l}, v_p(\det^{\frac{1}{2}} M(\alpha_{i_1,1})), \dots, v_p(\det^{\frac{1}{2}} M(\alpha_{i_1,i_1})), \\ & r_{i_1} + v_p(\det^{\frac{1}{2}} M(\alpha_{i_2,i_1+1})), \dots, r_{i_1} + v_p(\det^{\frac{1}{2}} M(\alpha_{i_2,i_2})), \\ & \dots, r_{i_1} + \dots + r_{i_{l-1}} + v_p(\det^{\frac{1}{2}} M(\alpha_{i_l,i_{l-1}+1})), \dots, r_{i_1} + \dots + r_{i_{l-1}} + v_p(\det^{\frac{1}{2}} M(\alpha_{i_l,i_l}))\}. \end{aligned} \quad (15)$$

Again, by expanding out the valuations of the determinants, we set

$$v_p(\det^{\frac{1}{2}}(M(\alpha_{i_j,k}))) = \min\{v_p(\hat{a}_1), v_p(\hat{a}_2), \dots, v_p(\hat{a}_{c_{\beta_{i_{j-1}}}}), v_p(a_{c_{\beta_{i_{j-1}+1}}}), \dots, v_p(a_{c_{\beta_k}})\}$$

for each $k \in \{i_{j-1} + 1, \dots, i_j\}$, where \hat{a}_n denotes the reduction mod $p^{r_{i_1} + \dots + r_{i_{j-1}}}$ as explained in Section 5, in particular in connection with equation (8). Now we can substitute the above expansions back inside (15), and cancel the redundant minima inside the expression. We can also cancel all the sums with reductions in them, as the minimum cannot be attained at these points because

$$r_{i_1} + \dots + r_{i_{j-1}} + v_p(\hat{a}_n) \geq v_p(a_n) \quad (16)$$

for any $n \in \{1, \dots, c_{\beta_{i_{j-1}}}\}$.

Let $\mathbf{r}_{i_1} + \mathbf{r}_{i_2} + \dots + \mathbf{r}_{i_{j-1}} + \mathbf{a}_{c_{\beta_{i_j}}}$ denote the $(c_{\beta_{i_j}} - c_{\beta_{i_{j-1}}})$ -tuple

$$(r_{i_1} + r_{i_2} + \dots + r_{i_{j-1}} + v_p(a_{c_{\beta_{i_{j-1}+1}}}), \dots, r_{i_1} + r_{i_2} + \dots + r_{i_{j-1}} + v_p(a_{c_{\beta_{i_j}}})) ,$$

where $c_{\beta_{i_j}}$ is the codimension of $F_{\beta_{i_j}}$ in $\mathbb{G}(i_j - 1, d' - 1)$. Then our weight function becomes

$$\begin{aligned} & w'(\Lambda', J_1, r_{J_1}) \\ & = d(r_{i_1} + \dots + r_{i_l}) \\ & - \min\{r_{i_1} + \dots + r_{i_l}, v_p(a_{c_{\beta_{i_1}}}), \mathbf{r}_{i_1} + \mathbf{a}_{c_{\beta_{i_2}}}, \mathbf{r}_{i_1} + \mathbf{r}_{i_2} + \mathbf{a}_{c_{\beta_{i_3}}}, \dots, \mathbf{r}_{i_1} + \dots + \mathbf{r}_{i_{l-1}} + \mathbf{a}_{c_{\beta_{i_l}}}\} \end{aligned}$$

where the highest dimensional linear subspace contained in the Pfaffian, or the dimension of the subspace in a flag where the corank of the augmented matrix of relations changes to 0, is $i_l - 1$. Recall Remark 6.7.

By applying Lemma 6.4, we deduce that the general weight function for a lattice Λ' of type $I = \{i_1, \dots, i_m\}_<$ is given by

$$\begin{aligned} w'(\Lambda', I, r_I) &= d(r_{i_1} + \dots + r_{i_m}) \\ &\quad - \min\{r_{i_1} + \dots + r_{i_l}, v_p(a_{c_{\beta_{i_1}}}), \mathbf{r}_{i_1} + \mathbf{a}_{c_{\beta_{i_2}}}, \mathbf{r}_{i_1} + \mathbf{r}_{i_2} + \mathbf{a}_{c_{\beta_{i_3}}}, \dots, \mathbf{r}_{i_1} + \dots + \mathbf{r}_{i_{l-1}} + \mathbf{a}_{c_{\beta_{i_l}}}\} \end{aligned}$$

where the highest dimensional linear subspace contained in the Pfaffian, or the dimension of the subspace in a flag where the corank of the augmented matrix of relations changes to 0, is $i_l - 1$. Recall Remark 6.7.

If $1 \in I$ then we have

$$\begin{aligned} w'(\Lambda', I, r_I) &= d(r_{i_1} + \dots + r_{i_m}) - \min\{r_1, v_p(a_1)\} \\ &\quad - \min\{r_{i_1} + \dots + r_{i_l}, v_p(a_{c_{\beta_{i_1}}}), \mathbf{r}_{i_1} + \mathbf{a}_{c_{\beta_{i_2}}}, \mathbf{r}_{i_1} + \mathbf{r}_{i_2} + \mathbf{a}_{c_{\beta_{i_3}}}, \dots, \mathbf{r}_{i_1} + \dots + \mathbf{r}_{i_{l-1}} + \mathbf{a}_{c_{\beta_{i_l}}}\}. \end{aligned}$$

8 Decomposing the zeta function

Now we can return to the main topic of the current paper. Recall that, we are interested in the normal zeta function

$$\zeta_G^\triangleleft(s) = \sum_{H \triangleleft_f G} |G : H|^{-s},$$

of a finitely generated, torsion-free, class-two-nilpotent group G . We shall now put the previous sections back into this context.

The zeta function admits the following Euler product decomposition

$$\zeta_G^\triangleleft(s) = \prod_{p \text{ prime}} \zeta_{G,p}^\triangleleft(s),$$

where $\zeta_{G,p}^\triangleleft(s)$ counts subgroups of finite p -power index only.

Let us again write $G = \Gamma \times Z^m$. Let \mathfrak{g} be the corresponding Lie ring constructed as an image of G under the log-map using the Mal'cev correspondence. Set $\mathfrak{g}_p := \mathfrak{g} \otimes \mathbb{Z}_p$. Then for almost all primes p we have from [12]

$$\zeta_{G,p}^\triangleleft(s) = \zeta_{\mathfrak{g}_p}^\triangleleft(s) = \zeta_p^\triangleleft(s).$$

Therefore, it suffices to count ideals in the associated Lie ring. Let \mathfrak{g}'_p denote the derived Lie ring.

Lemma 8.1. [12, Lemma 6.1] *Suppose $\mathfrak{g}'_p \cong \mathbb{Z}_p^{d+m}$ and $\mathfrak{g}'_p \cong \mathbb{Z}_p^{d'}$ as rings. For each lattice $\Lambda' \leq'_p \mathfrak{g}'_p$ put $X(\Lambda')/\Lambda' = Z(\mathfrak{g}'_p/\Lambda')$. Then*

$$\begin{aligned} \zeta_{\mathfrak{g}_p}^\triangleleft(s) &= \zeta_p^\triangleleft(s) = \zeta_{\mathbb{Z}_p^{d+m}}(s) \sum_{\Lambda' \leq'_p \mathfrak{g}'_p} |\mathfrak{g}'_p : \Lambda'|^{d+m-s} |\mathfrak{g}'_p : X(\Lambda')|^{-s} \\ &= \zeta_{\mathbb{Z}_p^{d+m}}(s) \zeta_p((d+d')s - d'(d+m)) A(p, p^{-s}), \end{aligned}$$

where

$$A(p, p^{-s}) = \sum_{\substack{\Lambda' \leq'_p \\ \Lambda' \text{ maximal}}} |'_p : \Lambda'|^{d+m-s} |_p : X(\Lambda')|^{-s}.$$

This lemma allows us to restrict to lattices in the centre only. We enumerate such lattices according to the elementary divisor type of the lattice, as explained in Section 4. It is enough to consider only maximal lattices of p -power index, since $\Lambda' = p^{r_{d'}} \Lambda'_{max}$ if Λ' is not maximal, where Λ'_{max} is maximal in its class. Now

$$|'_p : \Lambda'| = p^{d' r_{d'}} |'_p : \Lambda'_{max}|$$

and

$$|_p : X(\Lambda')| = p^{d' r_{d'}} |_p : X(\Lambda'_{max})|.$$

We define the weight functions

$$\begin{aligned} w(\Lambda') &:= \log_p(|'_p : \Lambda'|) \\ w'(\Lambda') &:= \log_p(|_p : X(\Lambda')|), \end{aligned}$$

where Λ' is a maximal lattice. Then

$$A(p, p^{-s}) = \sum_{\substack{\Lambda' \leq'_p \\ \Lambda' \text{ maximal}}} p^{(d+m)w(\Lambda') - s(w(\Lambda') + w'(\Lambda'))}.$$

Note that the $w'(\Lambda')$ is precisely the function we considered in Section 6, while $w(\Lambda')$ is easily read off from the type of the lattice. Indeed, if the type of Λ' is $\nu = (I, r_I)$, then

$$w(\Lambda') = \sum_{i \in I} i r_i.$$

9 Indexing

We can decompose the generating function $A(p, p^{-s})$ further to run over lattices of fixed flag type, and write it as

$$A(p, p^{-s}) = \sum_{I \subseteq \{1, \dots, d'-1\}} A_I(p, p^{-s}), \quad (17)$$

where

$$A_I(p, p^{-s}) = \sum_{\nu(\Lambda')=I} p^{(d+m)w(\Lambda') - s(w(\Lambda') + w'(\Lambda'))} \mu(\Lambda'). \quad (18)$$

Here Λ' is a representative lattice of flag type I and multiplicity $\mu(\Lambda')$, as in Proposition 4.3.

A more subtle decomposition is needed in order to reveal the dependence on the underlying geometry.

If the $(i_j - 1)$ -dimensional subspace of the flag of type $I = \{i_1, \dots, i_k\}_<$ is contained on the Pfaffian, we shall write i_j^* in the indexing set I to indicate this fact. For example

$$A_{2^*}(p, p^{-s}) = \sum_{\nu(\Lambda')=2^*} p^{(d+m)w(\Lambda')-s(w(\Lambda')+w'(\Lambda'))} \mu(\Lambda')$$

where the sum is taken over lattices Λ' of flag type $\{2\}$ such that their associated flag them consists only of a line which lies completely on the Pfaffian hypersurface. Thus with this notation we have $I = \{i_1, \dots, i_k\} \in \{1, \dots, d' - 1, 1^*, 2^*, \dots, l^*\}$, where l denotes the highest dimensional linear subspace on the Pfaffian.

The indexing and further decomposition of $A(p, p^{-s})$ are done via *admissible* subsets $I \subseteq \{1, \dots, d' - 1, 1^*, 2^*, \dots, l^*\}$. We call I admissible if the following conditions hold:

- (i) Only i_j or i_j^* can belong to I , but not both;
- (ii) If $i_j^* \in I$ then $i_k \notin I$ for all $k \leq j$.

Then we have

$$\begin{aligned} A(p, p^{-s}) = & \sum_{I \subseteq \{1, \dots, d'-1\}} c_{I,p} A_I(p, p^{-s}) + \sum_{\substack{I=1^* \cup J_2 \\ J_2 \subseteq \{2, \dots, d'-1\}}} c_{I,p} A_I(p, p^{-s}) + \\ & + \sum_{\substack{I=J_1 \cup 2^* \cup J_2 \\ J_1 \subseteq \{1^*\} \\ J_2 \subseteq \{3, \dots, d'-1\}}} c_{I,p} A_I(p, p^{-s}) + \dots + \sum_{\substack{I=J_1 \cup l^* \cup J_2 \\ J_1 \subseteq \{1^*, \dots, l-1^*\} \\ J_2 \subseteq \{l+1, \dots, d'-1\}}} c_{I,p} A_I(p, p^{-s}). \end{aligned}$$

It is possible to give an explicit description of the coefficients $c_{I,p}$. First let $\mathbf{n}_{\mathbb{B}_{i_j}}(p)$ denote the number of \mathbb{F}_p -points on the component $F_{\mathbb{B}_i}$ of $F_{i-1}(\mathfrak{P}_G)$, and let $\delta_{\mathbb{B}_i}$ be zero or one depending whether the corank of this component is zero or one. Also let $b_I(p)$ be the number of \mathbb{F}_p -points on the flag variety defined by lattices of flag type I . This is equal to

$$b_I(p) = \binom{d'}{i_l} \binom{i_l}{i_{l-1}} \dots \binom{i_3}{i_2} \binom{i_2}{i_1},$$

for $I = \{i_1, i_2, \dots, i_l\} \subseteq \{1, \dots, d'\}$. We define the flag type $I - k := \{i_j - k : i_j \in I\} \subseteq \{1, \dots, d' - k\}$. We claim that if $I = \{i_1, \dots, i_n\}$ then

$$c_{I,p} = b_I(p) - b_{I-i_1}(p) \left(\sum_{\mathbb{B}_{i_1}} \delta_{\mathbb{B}_i} \mathbf{n}_{\mathbb{B}_{i_1}}(p) \right), \quad (19)$$

while for $I = \{i_1^*, \dots, i_n^*, k^*, j_1, \dots, j_r\}$.

$$c_{I,p} = b_{J_2-k}(p) \left(\sum_{\mathbb{B}_k} \delta_{\mathbb{B}_i} \mathbf{n}_{\mathbb{B}_k}(p) \right) b_{J_1}(p) - b_{J_2-(k-j_1)}(p) \left(\sum_{\mathbb{B}_{j_1}} \delta_{\mathbb{B}_i} \mathbf{n}_{\mathbb{B}_{j_1}}(p) \right) b_{J_1 \cup k}(p). \quad (20)$$

This essentially comes from the formulae for the number of points on flag varieties. In (19) we have a flag variety where no part of the flag intersects the Pfaffian hypersurface, so at the level of (i_1-1) -spaces we need to subtract the number $\sum_{\mathbb{B}_{i_1}} \delta_{\mathbb{B}_i} \mathbf{n}_{\mathbb{B}_{i_1}}(p)$ of (i_1-1) -spaces on the Pfaffian, for which the weight function is different. The coefficient in (20) is derived in a similar fashion. As $(k-1)$ -dimensional spaces lie on the Pfaffian we need to compute, $\sum_{\mathbb{B}_k} \delta_{\mathbb{B}_i} \mathbf{n}_{\mathbb{B}_k}(p)$. Restricting to any $(k-1)$ -dimensional space of these it is clear that the flag variety with the highest subspace being this fixed $(k-1)$ -dimensional spaces, lies completely on the Pfaffian. In higher dimensions we count only those spaces that are off the Pfaffian at the level j_1 , and thus we subtract $\sum_{\mathbb{B}_{j_1}} \delta_{\mathbb{B}_i} \mathbf{n}_{\mathbb{B}_{j_1}}(p)$.

Rearranging such that we can pull out the coefficients $\mathbf{n}_{\mathbb{B}_{i_j}}(p)$, we obtain the following decomposition

$$A(p, p^{-s}) = W_0(p, p^{-s}) + \sum_{i=1}^l \sum_{\mathbb{B}_i} \delta_{\mathbb{B}_i} \mathbf{n}_{\mathbb{B}_i}(p) W_{\mathbb{B}_i}(p, p^{-s})$$

where

$$W_0(p, p^{-s}) = \sum_{I \subseteq \{1, \dots, d'-1\}} b_I(p) A_I(p, p^{-s})$$

and

$$\begin{aligned} W_{\mathbb{B}_i}(p, p^{-s}) &= \sum_{\substack{I=J_1 \cup i^* \cup J_2 \\ J_1 \subseteq \{1^*, 2^*, \dots, i-1^*\} \\ J_2 \subseteq \{i+1, \dots, d'-1\}}} b_{J_2-i}(p) b_{J_1}(p) A_I(p, p^{-s}) - \sum_{\substack{I=J_1 \cup i \cup J_2 \\ J_1 \subseteq \{1^*, 2^*, \dots, i-1^*\} \\ J_2 \subseteq \{i+1, \dots, d'-1\}}} b_{J_2-i}(p) b_{J_1}(p) A_I(p, p^{-s}) \\ &= \sum_{\substack{J_1 \subseteq \{1^*, 2^*, \dots, i-1^*\} \\ J_2 \subseteq \{i+1, \dots, d'-1\}}} b_{J_2-i}(p) b_{J_1}(p) (A_{J_1 \cup i^* \cup J_2}(p, p^{-s}) - A_{J_1 \cup i \cup J_2}(p, p^{-s})). \end{aligned}$$

Here i^* is a lattice that gives a point on $F_{\mathbb{B}_i}$, and J_1 runs over all lattices that are contained in this particular component $F_{\mathbb{B}_i}$.

10 Igusa factors

Now that we have an expression for the generating function, and all the terms appearing in it, the rest of the proof of Theorem 1.7 is basically a generalisation of the summation formulae that appear in [17], see Lemmas 4.2 to 4.16, in particular.

Lemma 10.1.

$$\begin{aligned}
W_0(p, p^{-s}) &= \sum_{I \subseteq \{1, \dots, d'-1\}} b_I(p) A_I(p, p^{-s}) \\
&= \sum_{I \subseteq \{1, \dots, d'-1\}} b_I(p^{-1}) \prod_{i \in I} \frac{X_i}{1 - X_i} \\
&= I_{d'-1}(X_1, \dots, X_{d'-1}),
\end{aligned}$$

where $X_i = p^{(d+d'+m-i)i-(d+i)s}$.

Proof. Using the Lemma 4.3 we can write

$$\begin{aligned}
&\sum_{I \subseteq \{1, \dots, d'-1\}} b_I(p) A_I(p, p^{-s}) \\
&= \sum_{I \subseteq \{1, \dots, d'-1\}} b_I(p) \prod_{i \in I} \sum_{r_i=1}^{\infty} p^{(d+m)ir_i-(d+i)r_is} p^{-\dim \mathcal{F}_I} p^{(d'-i)ir_i} \\
&= \sum_{I \subseteq \{1, \dots, d'-1\}} b_I(p^{-1}) \prod_{i \in I} \sum_{r_i=1}^{\infty} p^{i(d+d'+m-i)r_i-(d+i)r_is} \\
&= \sum_{I \subseteq \{1, \dots, d'-1\}} b_I(p^{-1}) \prod_{i \in I} \frac{p^{i(d+d'+m-i)-(d+i)s}}{1 - p^{i(d+d'+m-i)-(d+i)s}}.
\end{aligned}$$

□

Lemma 10.2.

$$\begin{aligned}
&W_{\mathbb{B}_i}(p, p^{-s}) \\
&= \sum_{\substack{J_1 \subseteq \{1^*, 2^*, \dots, i-1^*\} \\ J_2 \subseteq \{i+1, \dots, d'-1\}}} b_{J_2-i}(p) b_{J_1}(p) (A_{J_1 \cup i^* \cup J_2}(p, p^{-s}) - A_{J_1 \cup i \cup J_2}(p, p^{-s})) \\
&= \sum_{J_2 \subseteq \{i+1, \dots, d'-1\}} b_{J_2-i}(p^{-1}) \prod_{j_2 \in J_2} \frac{X_{j_2}}{1 - X_{j_2}} \sum_{J_1 \subseteq \{1^*, 2^*, \dots, i-1^*\}} b_{J_1}(p) (A_{J_1 \cup i^*}(p, p^{-s}) - A_{J_1 \cup i}(p, p^{-s})) \\
&= I_{d'-i-1}(X_{i+1}, \dots, X_{d'-1}) \sum_{J_1 \subseteq \{1^*, 2^*, \dots, i-1^*\}} b_{J_1}(p) (A_{J_1 \cup i^*}(p, p^{-s}) - A_{J_1 \cup i}(p, p^{-s})),
\end{aligned}$$

where $X_i = p^{(d+d'+m-i)i-(d+i)s}$. Again i^* is a lattice that gives a point on $F_{\mathbb{B}_i}$, and J_1 runs over all lattices that are contained on this particular component.

Lemma 10.3.

$$\begin{aligned}
&\sum_{J_1 \subseteq \{1^*, 2^*, \dots, i-1^*\}} b_{J_1}(p) (A_{J_1 \cup i^*}(p, p^{-s}) - A_{J_1 \cup i}(p, p^{-s})) \\
&= \sum_{J_1 \subseteq \{1^*, 2^*, \dots, i-1^*\}} b_{J_1}(p^{-1}) \prod_{j_1 \in J_1} \frac{Y_{\mathbb{B}_{j_1}}}{1 - Y_{\mathbb{B}_{j_1}}} (A_{i^*}(p, p^{-s}) - A_i(p, p^{-s})) \\
&= I_{i-1}(Y_1, \dots, Y_{\mathbb{B}_{i-1}}) (A_{i^*}(p, p^{-s}) - A_i(p, p^{-s})),
\end{aligned}$$

where $Y_{\beta_i} = p^{i(d+m)+c_{\beta_i}-(d+i-1)s}$ for $\beta_1 > 1$, while $Y_1 = p^{d+m+c_1-(d-1)s}$.

It remains to calculate $A_{i^*}(p, p^{-s}) - A_i(p, p^{-s})$ and justify the terms X_i and Y_{β_i} appearing in the formulae above.

Proposition 10.4. *Let $G = \Gamma \times \mathbb{Z}^m$ as before. Let $d = h(G/Z(G))$, and $d' = h([G, G])$. Assume d is even. Let n_i be the dimension of $\mathbb{G}(i-1, d'-1)$, d_{β_i} the dimension of F_{β_i} and c_{β_i} its codimension, so $n_i = c_{\beta_i} + d_{\beta_i}$. Then*

$$\begin{aligned} A_{i^*} - A_i &= \sum_{r_i=1}^{\infty} \sum_{a_1=1}^{r_i} \sum_{a_2=1}^{r_i} \cdots \sum_{a_{n_i}=1}^{r_i} \mu(r_i, a_1, a_2, \dots, a_{n_i}) p^{i(d+m)r_i-(d+i)r_i s} (p^{st_i \min\{r_i, a_1, a_2, \dots, a_{n_i}\}} - 1) \\ &= \frac{p^{i(d+m)-(d+i-t_i)s} (1 - p^{-st_i})}{(1 - p^{i(d+m)+d_{\beta_i}-(d+i-t_i)s}) (1 - p^{i(d+m)+n_i-(d+i)s})} = \frac{p^{-d_{\beta_i}} Y_{\beta_i} - p^{-n_i} X_i}{(1 - Y_{\beta_i})(1 - X_i)}, \end{aligned}$$

where $X_i = p^{i(d+d'+m-i)-(d+i)s}$ and $Y_{\beta_i} = p^{(i(d+m)+d_{\beta_i})-(d+i-t_i)s}$. Furthermore, $t_1 = 2$ and $t_i = 1$ for $i \geq 2$.

Proof. The proof is given in Proposition 4.12 in [17] and is essentially a manipulation of infinite series. \square

We conclude that the $W_{\beta_i}(p, p^{-s})$ are of the form given in Theorem 1.7. This concludes the proof of Theorem 1.7.

11 Example

In this section show an explicit application of the main theorem, by defining a class two Lie ring (recall, that there is also a class two nilpotent group with the same presentation) whose Pfaffian is the quadric Segre surface in \mathbb{P}^3 .

Let $G_{\mathcal{S}}$ have the presentation

$$G_{\mathcal{S}} = \langle x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 : [x_1, x_3] = y_1, [x_1, x_4] = y_2, [x_2, x_3] = y_3, [x_2, x_4] = y_4 \rangle.$$

The matrix of relations $M(\mathbf{y})_{ij} = [x_i, x_j]$, is

$$M(\mathbf{y}) = \begin{pmatrix} 0 & 0 & y_1 & y_2 \\ 0 & 0 & y_3 & y_4 \\ -y_1 & -y_3 & 0 & 0 \\ -y_2 & -y_4 & 0 & 0 \end{pmatrix}$$

and the Pfaffian is thus defined by

$$\mathcal{S} : y_1 y_4 - y_2 y_3 = 0.$$

Since $\mathcal{S} \cong \mathbb{P}^1 \times \mathbb{P}^1$, the number of \mathbb{F}_p -rational points on \mathcal{S} is $|\mathcal{S}(\mathbb{F}_p)| = (p+1)^2$. Furthermore, over \mathbb{F}_p there are $2(p+1)$ lines on this surface and no higher dimensional linear subspaces, see e.g. [14].

Note that in this group $Z(G_{\mathcal{S}}) = [G_{\mathcal{S}}, G_{\mathcal{S}}]$, so in the application of the Theorem 1.7, we have $m = 0$.

Theorem 11.1. *For almost all primes p , the local normal zeta function of G_S is given by*

$$\zeta_{G_S,p}^{\triangleleft}(s) = \zeta_{\mathbb{Z}^4,p}(s) \cdot \zeta_p(8s-16) \cdot (W_0(p, p^{-s}) + (p+1)^2 W_1(p, p^{-s}) + 2(p+1) W_2(p, p^{-s}))$$

where

$$\begin{aligned} W_0(p, T) &= I_3(X_1, X_2, X_3) \\ W_1(p, T) &= I_2(X_2, X_3) E_1(X_1, Y_1) \\ W_2(p, T) &= I_1(X_3) E_2(X_2, Y_2) I_1(Y_1) \end{aligned}$$

with $X_i = p^{i(8-i)-(4+i)s}$, $Y_1 = p^{6-3s}$ and $Y_2 = p^{9-5s}$.

11.1 Calculation

As before, the basic building blocks of the zeta function are the generating functions over lattices of fixed elementary divisor types.

$$A_I(p, T) = \sum_{\nu(\Lambda')=I} p^{dw(\Lambda')-s(w(\Lambda')+w'(\Lambda'))} \mu(\Lambda').$$

The elementary divisor types of the lattice Λ' in this case are

$$(p^{r_1+r_2+r_3}, p^{r_2+r_3}, p^{r_3}, 1),$$

where $r_i \geq 0$, the flag type is $I := \{i : r_i > 0\} \subseteq \{1, 2, 3\}$. Immediately it follows that $w(\Lambda') = \sum_{i \in I} i r_i$, and the multiplicity $\mu(\Lambda')$ can be calculated using the 1-1 correspondence between lattices and cosets.

Example 11.2. Let Λ' have elementary divisors $(p^{r_1+r_2}, p^{r_2}, 1, 1)$ so that type consists of $I = \{1, 2\}$ and $r_I = (r_1, r_2, 0, 0)$. The stabiliser of Λ' takes the form

$$\begin{pmatrix} \text{GL}_1(\mathbb{Z}_p) & * & * \\ p^{r_1} \mathbb{Z}_p & \text{GL}_1(\mathbb{Z}_p) & * \\ p^{r_1+r_2} \mathbb{Z}_p & p^{r_2} \mathbb{Z}_p & \text{GL}_2(\mathbb{Z}_p) \end{pmatrix}.$$

If we then exclude the terms coming from the flag varieties, a coset of the stabiliser takes form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ b_3 & 1 & 0 & 0 \\ b_2 & a_2 & 1 & 0 \\ b_1 & a_1 & 0 & 1 \end{pmatrix},$$

where $a_1, a_2 \in p\mathbb{Z}_p/(p^{r_2})$, $b_1, b_2 \in p\mathbb{Z}_p/(p^{r_1+r_2})$, $b_3 \in p\mathbb{Z}_p/(p^{r_1})$. We need to calculate how many choices we can make for each a_i, b_i . Thus the multiplicity of Λ' is

$$\begin{aligned} \mu(1, 2) &= \sum \mu(r_2; b_2) \mu(r_2, b_1) \mu(r_1 + r_2; a_3) \mu(r_2 + r_1; a_2) \mu(r_1; a_1) \\ &= p^{-5} p^{3r_1+4r_2} \end{aligned}$$

where the a_i, b_i are variables, and the ranges of summations are the obvious ones from equation (5).

For $w'(\Lambda')$ we need to consider the flags associated with Λ' , defined in Section 5. Some of the flags in \mathbb{F}_p^4 intersect with the Segre surface and in the light of the congruence conditions, the weight function $w'(\Lambda')$ changes.

As in (9) Section 6, in order to determine $w'(\Lambda')$ we want to understand the different solutions to the set of congruences

$$\begin{aligned} \mathbf{g}M(\beta_1) &\equiv 0 \pmod{p^{r_1+r_2+r_3}} \\ \mathbf{g}M(\beta_2) &\equiv 0 \pmod{p^{r_2+r_3}} \\ \mathbf{g}M(\beta_3) &\equiv 0 \pmod{p^{r_3}}. \end{aligned}$$

Note that we only insist that $r_i \geq 0$.

Since the Segre surface does not contain any planes, the vector β_3 can always be chosen such that $\det^{\frac{1}{2}}(M(\beta_3))$ is a p -adic unit and the third condition reduces to

$$\mathbf{g} \equiv 0 \pmod{p^{r_3}},$$

and we get an immediate reduction, as in Lemma 6.4,

$$\begin{aligned} w'(1, 3) &= w'(1) + w'(3) \\ w'(2, 3) &= w'(2) + w'(3) \\ w'(1, 2, 3) &= w'(1, 2) + w'(3). \end{aligned} \tag{21}$$

This induces a similar reduction in generating functions.

$$\begin{aligned} A_{1^*,3} &= p^2 A_{1^*} \cdot A_3 \\ A_{1^*,2,3} &= p^2 A_{1^*,2} \cdot A_3 \\ A_{2^*,3} &= p^2 A_{2^*} \cdot A_3 \\ A_{1^*,2^*,3} &= p^2 A_{1^*,2^*} \cdot A_3, \end{aligned} \tag{22}$$

because $w(J \cup 3) = w(3) + w(J)$, for $J \subseteq \{1^*, 2^*, 1, 2\}$ and the μ -function has enough multiplicative properties, e.g., $\mu(r_i + r_3; a_1) = p^{r_3} \mu(r_i; a_1)$.

There are four additional cases we need to consider reflecting the four different geometric configurations how a flag can intersect the Segre surface.

1. $M(\beta_1)$ and $M(\beta_2)$ are both non-singular mod p .
2. $M(\beta_1)$ is singular mod p , but $M(\beta_2)$ is not singular mod p , and the line $\langle \overline{\beta_1}, \overline{\beta_2} \rangle$ does not lie on the Segre surface.
3. $M(\beta_1)$ and $M(\beta_2)$ are both singular mod p , and the line $\langle \overline{\beta_1}, \overline{\beta_2} \rangle$ lies on the Segre surface.
4. $M(\beta_1)$ and $M(\beta_2)$ are both singular mod p , and the flag $\langle \overline{\beta_1} \rangle < \langle \overline{\beta_1}, \overline{\beta_2} \rangle$ consisting of a point and a line lies on the Segre surface.

We are now justified to use the same indexing as in Section 9 and if we denote by $\mathbf{n}_1(p)$ the number of points on the Pfaffian, and similarly by $\mathbf{n}_2(p)$ the number of lines, the generating function takes the explicit form as a sum over all admissible subsets of $\{1^*, 2^*, 1, 2, 3\}$:

$$\begin{aligned}
A(p, p^{-s}) = & A_\emptyset + \left(\binom{4}{1}_p - \mathbf{n}_1(p) \right) A_1 + \left(\binom{4}{2}_p - \mathbf{n}_2(p) \right) A_2 \\
& + \binom{4}{3}_p A_3 + \binom{3}{1}_p \left(\binom{4}{1}_p - \mathbf{n}_1(p) \right) A_{1,2} \\
& + \binom{3}{2}_p \left(\binom{4}{1}_p - \mathbf{n}_1(p) \right) A_{1,3} \\
& + \binom{2}{1}_p \left(\binom{4}{2}_p - \mathbf{n}_2(p) \right) A_{2,3} + \binom{2}{1}_p \binom{3}{1}_p \left(\binom{4}{1}_p - \mathbf{n}_1(p) \right) A_{1,2,3} \\
& + \mathbf{n}_1(p) A_{1^*} + \left(\binom{3}{1}_p \mathbf{n}_1(p) - \mathbf{n}_2(p) \binom{2}{1}_p \right) A_{1^*,2} + \binom{3}{2}_p \mathbf{n}_1(p) A_{1^*,3} + \\
& \binom{2}{1}_p \left(\binom{3}{1}_p \mathbf{n}_1(p) - \mathbf{n}_2(p) \binom{2}{1}_p \right) A_{1^*,2,3} + \mathbf{n}_2(p) A_{2^*} \\
& + \mathbf{n}_2(p) \binom{2}{1}_p A_{1^*,2^*} + \binom{2}{1}_p \mathbf{n}_2(p) A_{2^*,3} + \binom{2}{1}_p \mathbf{n}_2(p) \binom{2}{1}_p A_{1^*,2^*,3}.
\end{aligned}$$

We rearrange this sum in order to see exactly which parts depend on $\mathbf{n}_1(p)$ and $\mathbf{n}_2(p)$, and obtain

$$A(p, p^{-s}) = W_0(p, p^{-s}) + \mathbf{n}_1(p) W_1(p, p^{-s}) + \mathbf{n}_2(p) W_2(p, p^{-s}), \quad (23)$$

where

$$\begin{aligned}
W_0(p, p^{-s}) &= \sum_{I \subseteq \{1,2,3\}} b_I(p) A_I(p, p^{-s}), \\
W_1(p, p^{-s}) &= \sum_{I \subseteq \{2,3\}} b_I(p) (A_{1^* \cup I}(p, p^{-s}) - A_{1 \cup I}(p, p^{-s})) \\
W_2(p, p^{-s}) &= \left(1 + \binom{2}{1}_p p^2 A_3 \right) \left((A_{2^*} - A_2) + \binom{2}{1}_p (A_{1^*,2^*} - A_{1^*,2}) \right).
\end{aligned}$$

The third formula follows from the reductions in generating functions (22). By Lemma 10.1 $W_0(p, p^{-s}) = I_3(X_1, X_2, X_3)$, as wanted.

Now we need to determine the generating functions $A_{1^*} - A_1$, $A_{2^*} - A_2$ and $A_{1^*,2^*} - A_{1^*,2}$, which will finish this example.

The easiest case is $A_{1^*} - A_1$. By Lemma 6.5 the lattices $(p^{r_1}, 1, 1, 1)$ lifting a point on Pfaffian are in 1-1 correspondence with the vector $\beta_1 =$

$(1, a_1, a_2, a_3)^t$, $a_i \in p\mathbb{Z}_p/p^{r_1}$, and the weight function is $w'(\Lambda') = 4r_1 - 2\min\{r_1, v_p(\det^{\frac{1}{2}}M(\beta_1))\}$. By Lemma 7.1 on p -adic valuations, this is equal to $w'(\Lambda') = 4r_1 - 2\min\{r_1, v_p(a_1)\}$ and hence we may apply Proposition 10.4 and get

$$A_{1*} - A_1 = \frac{p^{4-3s} - p^{5-5s}}{(1 - p^{5-3s})(1 - p^{7-5s})} = E_1(X_1, Y_1),$$

where $X_1 = p^{7-5s}$ and $Y_1 = p^{5-3s}$.

Next we calculate the weight function over lattices that lift lines on the Segre surface. There are two rulings of lines on the Segre surface, but in fact both of them behave similarly. We consider the line defined by $y_1 = y_2 = 0$. Lattices Λ' of type $\{2^*\}$ lifting this line are in one-to-one correspondence with pairs of vectors $\beta_1 = (a_1, a_2, 0, 1)^t, \beta_2 = (b_1, b_2, 1, 0)^t$ where $a_i, b_i \in p\mathbb{Z}/(p^{r_2})$. We will define an equivalent lattice as in Definition 5.2

$$\alpha_1 = \begin{pmatrix} \lambda_1 a_1 \\ \lambda_1 a_2 \\ 0 \\ \lambda_1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} \lambda_1 a_1 + \lambda_2 b_1 \\ \lambda_1 a_2 + \lambda_2 b_2 \\ \lambda_2 \\ \lambda_1 \end{pmatrix}.$$

Now the congruences to be satisfied by \mathbf{g} are

$$\mathbf{g} \begin{pmatrix} 0 & 0 & \lambda_1 a_1 & \lambda_1 a_2 \\ 0 & 0 & 0 & \lambda_1 \\ -\lambda_1 a_1 & 0 & 0 & 0 \\ -\lambda_1 a_2 & -\lambda_1 & 0 & 0 \end{pmatrix} \equiv 0 \pmod{p^{r_2}}$$

$$\mathbf{g} \begin{pmatrix} 0 & 0 & \lambda_1 a_1 + \lambda_2 b_1 & \lambda_1 a_2 + \lambda_2 b_2 \\ 0 & 0 & \lambda_2 & \lambda_1 \\ -\lambda_1 a_1 - \lambda_2 b_1 & -\lambda_2 & 0 & 0 \\ -\lambda_1 a_2 - \lambda_2 b_2 & -\lambda_1 & 0 & 0 \end{pmatrix} \equiv 0 \pmod{p^{r_2}}.$$

The corank of this system of linear equations is 1, as suitable row and column operations show. Thus we get the result as in the equation (11) the weight function is $w'(\Lambda') = 4r_2 - \min\{r_2, v_p(\det^{\frac{1}{2}}(M(\alpha_1))), v_p(\det^{\frac{1}{2}}(M(\alpha_2)))\}$. Now applying the reduction Lemma 7 in the p -adic valuations, we have to evaluate $\min\{r_2, v_p(\lambda_1^2 a_1), v_p(\lambda_1^2 a_1 + \lambda_1 \lambda_2 (b_1 - a_2) - \lambda_1 \lambda_2 b_2)\}$. Since λ_1, λ_2 are p -adic units, this reduces to $\min\{r_2, v_p(a_1), v_p(b_1), v_p(b_2)\}$. By Proposition 10.4

$$A_{2*} - A_2 = \sum_{r_2=1}^{\infty} \sum_{a_i, b_i=1}^{r_2} p^{8r_2} T^{6r_2 - \min\{r_2, b_1, b_2, a_1\}} \mu(r_2, a_1, a_2, b_1, b_2)$$

$$= \frac{p^{8-5s}(1 - p^{9-6s})}{(1 - p^{9-5s})(1 - p^{12-6s})} = E_2(X_2, Y_2),$$

for $X_2 = p^{12-6s}$ and $Y_2 = p^{9-5s}$.

Finally, we need to calculate the generating function $A_{1^*,2^*}$ over a lattice of type $(p^{r_1+r_2}, p^{r_2}, 1, 1)$. A lattice Λ' lifting a flag of type $\{1^*, 2^*\}$, is in one-to-one correspondence with pairs of vectors

$$\beta_1 = (a_1, a_2, a_3, 1)^t$$

$$\beta_2 = (b_1, b_2, 1, 0)^t$$

where $a_1, a_2 \in p\mathbb{Z}/(p^{r_1+r_2})$, $a_3 \in p\mathbb{Z}/(p^{r_1})$ and $b_1, b_2 \in p\mathbb{Z}/(p^{r_2})$. Any other choice of a flag and its affine neighbourhood behaves in the same way. Again, change variables

$$\alpha_1 = \begin{pmatrix} \lambda_1 a_1 \\ \lambda_1 a_2 \\ \lambda_1 a_3 \\ \lambda_1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} \lambda_1 \bar{a}_1 + \lambda_2 b_1 \\ \lambda_1 \bar{a}_2 + \lambda_2 b_2 \\ \lambda_1 \bar{a}_3 + \lambda_2 \\ \lambda_1 \end{pmatrix},$$

where \bar{a}_i denotes the reduction mod p^{r_2} . As in the previous case, the corank of the system of equations is 1, and thus using the equation (13) the weight function is

$$w'(\Lambda') = 4r_2 + 4r_1 - \min\{r_1, v_p(\det^{\frac{1}{2}}(M(\alpha_1)))\} - \min\{r_1 + r_2, v_p(\det^{\frac{1}{2}}(M(\alpha_1))), r_1 + v_p(\det^{\frac{1}{2}}(M(\alpha_2)))\},$$

and an application of Lemma 7 yields

$$\min\{r_1, v_p(\lambda_1^2(a_1 - a_2 a_3))\} = \min\{r_1, v_p(a_1)\}$$

and

$$\begin{aligned} & \min\{r_1 + r_2, v_p(\lambda_1^2(a_1 - a_2 a_3)), r_1 + v_p(\lambda_1^2(\bar{a}_1 - \bar{a}_2 \bar{a}_3) + \lambda_1 \lambda_2(b_1 - \bar{a}_2 - b_2 \bar{a}_3) + \lambda_2^2(b_2))\} \\ &= \min\{r_1 + r_2, v_p(a_1 - a_2 a_3), r_1 + \min\{v_p(\bar{a}_1 - \bar{a}_2 \bar{a}_3), v_p(b_1 - \bar{a}_2 - b_2 \bar{a}_3), v_p(b_2)\}\} \\ &= \min\{r_1 + r_2, v_p(a_1 - a_2 a_3), \min\{r_1 + v_p(\bar{a}_1 - \bar{a}_2 \bar{a}_3), r_1 + v_p(b_1 - \bar{a}_2 - b_2 \bar{a}_3), r_1 + v_p(b_2)\}\} \\ &= \min\{r_1 + r_2, v_p(a_1 - a_2 a_3), r_1 + v_p(\bar{a}_1 - \bar{a}_2 \bar{a}_3), r_1 + v_p(b_1 - \bar{a}_2 - b_2 \bar{a}_3), r_1 + v_p(b_2)\} \\ &= \min\{r_1 + r_2, v_p(a_1), r_1 + v_p(b_1), r_1 + v_p(b_2)\}. \end{aligned}$$

Now we can write $A_{1^*,2^*} - A_{1^*,2}$ in a form where we can first apply Lemma 10.3 to get

$$A_{1^*,2^*} - A_{1^*,2} = I_1(Y_1)(A_{2^*} - A_2)$$

and after that Proposition 10.4, to obtain the final result as above.

11.2 A closed expression

$$\zeta_{GS,p}^{\triangleleft}(s) = \zeta_{\mathbb{Z}^4,p}(s) \zeta_p(8s-16) \zeta_p(5s-7) \zeta_p(6s-12) \zeta_p(7s-15) \zeta_p(3s-6) \zeta_p(9s-5) \cdot W(p, p^{-s}),$$

where

$$\begin{aligned}
W(p, T) = & 1 + p^4T^4 + 2p^5T^3 - p^5T^5 + 2p^8T^5 + p^9T^5 - p^8T^6 - p^9T^6 + 2p^{10}T^6 \\
& + p^{11}T^7 + p^{12}T^7 + p^{13}T^7 + p^{14}T^7 - p^{10}T^8 - p^{11}T^8 - p^{12}T^8 + p^{13}T^8 \\
& - 2p^{13}T^9 - 2p^{14}T^9 + 2p^{15}T^9 + p^{16}T^9 + p^{14}T^{10} - 2p^{15}T^{10} - 2p^{16}T^{10} \\
& + p^{17}T^{10} + 2p^{18}T^{10} + 2p^{19}T^{10} - 2p^{17}T^{11} - p^{18}T^{11} - 2p^{19}T^{11} - p^{20}T^{11} \\
& - p^{18}T^{12} - 2p^{19}T^{12} - p^{21}T^{12} + p^{22}T^{12} + p^{23}T^{12} + p^{19}T^{13} - p^{20}T^{13} \\
& - p^{21}T^{13} - p^{22}T^{13} - p^{23}T^{13} + p^{24}T^{13} + p^{20}T^{14} + p^{21}T^{14} - p^{22}T^{14} \\
& - 2p^{24}T^{14} - p^{25}T^{14} - p^{23}T^{15} - 2p^{24}T^{15} - p^{25}T^{15} - 2p^{26}T^{15} + 2p^{24}T^{16} \\
& + 2p^{25}T^{16} + p^{26}T^{16} - 2p^{27}T^{16} - 2p^{28}T^{16} + p^{29}T^{16} + p^{27}T^{17} + 2p^{28}T^{17} \\
& - 2p^{29}T^{17} - 2p^{30}T^{17} + p^{30}T^{18} - p^{31}T^{18} - p^{32}T^{18} - p^{33}T^{18} + p^{29}T^{19} \\
& + p^{30}T^{19} + p^{31}T^{19} + p^{32}T^{20} + 2p^{33}T^{20} - p^{34}T^{20} - p^{35}T^{20} + p^{34}T^{21} \\
& + 2p^{35}T^{21} - p^{38}T^{21} + 2p^{38}T^{23} + p^{39}T^{23} + p^{43}T^{26}.
\end{aligned}$$

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